AB

INSA | INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Math Test #3 — 2h

March 31, 2016

No documents, no calculators, no cell phones or electronic devices allowed but you may keep your pet blobfish for moral support.

All your answers must be fully justified, unless noted otherwise.

For your convenience, we have recalled the main theorems of integral calculus at the end of this document.

Exercise 1. For each of the following power series, determine its radius of convergence R and determine an explicit expression (without the $\sum$ symbol) for $x \in (-R, R)$.

$$(1)\ \sum_{n=0}^{+\infty} \cosh(n) x^n, \qquad (2)\ \sum_{n=0}^{+\infty} (-1)^n \frac{n}{(2n+1)!} x^{2n}.$$

Exercise 2. Use the integral comparison test to determine the nature of the series

$$\text{(S)} \qquad \sum_n \frac{1}{n\ln(n)}$$

and to find an equivalent of the partial sum

$$S_N = \sum_{n=2}^{N} \frac{1}{n\ln(n)}$$

as $N \to +\infty$.

Exercise 3. We define the sequence $(u_n)_{n\geq 1}$ as

$$\forall n \in \mathbb{N}^*,\ u_n = -\ln(n) + \sum_{k=1}^{n} \frac{1}{k}.$$

1. Show that the series $\sum_n (u_{n+1} - u_n)$ converges.
2. Deduce that the the sequence $(u_n)_{n\geq 1}$ converges; we denote by γ the value of its limit.[1]
3. Show that the series

$$\text{(S)} \qquad \sum_n \left(\frac{1}{n} - \ln\left(\frac{n}{n-1}\right)\right)$$

converges and determine the value of the sum

$$\sum_{n=2}^{+\infty} \left(\frac{1}{n} - \ln\left(\frac{n}{n-1}\right)\right)$$

in terms of γ. Hint: express the partial sum of this series in terms of the sequence $(u_n)_{n\geq 1}$.

Exercise 4. Show that the series

$$\text{(S)} \qquad \sum_n \frac{(-1)^n}{1+\sqrt{n}}$$

converges. Consider the partial sums of the series (S):

$$\forall N \in \mathbb{N},\ S_N = \sum_{n=0}^{N} \frac{(-1)^n}{1+\sqrt{n}}.$$

Find a value of $N \in \mathbb{N}$ for which S_N is an approximation of the sum of the series (S) with error less than 10^{-5}.

[1] the number γ is known as the *Euler–Mascheroni constant*.

Exercise 5.

1. Show that for all $a, b \in \mathbb{R}_+^*$ the improper integral
$$I(a,b) = \int_0^{+\infty} \frac{e^{-at} - e^{-bt}}{t}\, dt$$
converges.

2. We define the function
$$\begin{array}{rccl} \varphi : & \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & I(1,x) = \displaystyle\int_0^{+\infty} \frac{e^{-t} - e^{-xt}}{t}\, dt. \end{array}$$
Show that φ is differentiable on $\mathbb{R}_+^*$ and, for $x \in \mathbb{R}_+^*$, find an explicit expression of $\varphi'(x)$.

3. Deduce an explicit expression of φ.

4. Let $a, b \in \mathbb{R}_+^*$. Use an appropriate substitution to compute the value of $I(a, b)$.