

**Exercise 1.**

1. Let  $x \in \mathbb{C}^*$  and define

$$\forall n \in \mathbb{N}, u_n = \cosh(n)|x|^n > 0.$$

We use the ratio test for the convergence of the series  $\sum_n u_n$ :

$$\frac{u_{n+1}}{u_n} = \frac{\cosh(n+1)}{\cosh(n)} |x| = \frac{e^{n+1} - e^{-n-1}}{e^n - e^{-n}} |x| \underset{n \rightarrow +\infty}{\sim} \frac{e^{n+1}}{e^n} |x| = e|x| \underset{n \rightarrow +\infty}{\rightarrow} e|x|.$$

Hence, by the ratio test, the series  $\sum_n u_n$  converges if  $|x| < 1/e$  and diverges if  $|x| > 1/e$ ; we conclude that  $R = 1/e$ .

Now, for  $x \in (-R, R)$ ,

$$\begin{aligned} \sum_{n=0}^{+\infty} \cosh(n)x^n &= \sum_{n=0}^{+\infty} \frac{e^n + e^{-n}}{2} x^n = \frac{1}{2} \sum_{n=0}^{+\infty} (ex)^n + \left(\frac{x}{e}\right)^n \\ &= \frac{1}{2} \left( \sum_{n=0}^{+\infty} (ex)^n + \sum_{n=0}^{+\infty} \left(\frac{x}{e}\right)^n \right) \quad \text{since both series converge} \\ &= \frac{1}{2} \left( \frac{1}{1-ex} + \frac{1}{1-x/e} \right) \end{aligned}$$

2. Let  $x \in \mathbb{C}^*$  and define

$$\forall n \in \mathbb{N}, u_n = \frac{n}{(2n+1)!} |x|^{2n} > 0.$$

We use the ratio test for the convergence of the series  $\sum_n u_n$ :

$$\frac{u_{n+1}}{u_n} = \frac{n+1}{(2n+3)!} |x|^{2n+2} = \frac{n+1}{(2n+2)(2n+3)n} |x|^2 \underset{n \rightarrow +\infty}{\rightarrow} 0.$$

Hence, by the ratio test, the series  $\sum_n u_n$  converges. We conclude that  $R = +\infty$ .

For  $x \in \mathbb{R}^*$ ,

$$\begin{aligned} \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} x^{2n} &= \frac{1}{2} \sum_{n=0}^{+\infty} \frac{(-1)^n (2n+1) - 1}{(2n+1)!} x^{2n} = \frac{1}{2} \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} \left( \frac{1}{(2n)!} - \frac{1}{(2n+1)!} \right) x^{2n} \\ &= \frac{1}{2} \left( \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n)!} x^{2n} - \frac{1}{x} \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \right) \quad \text{since both series converge} \\ &= \frac{1}{2} \cos(x) - \frac{1}{2x} \sin(x). \end{aligned}$$

If  $x = 0$ , we obtain:

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} 0^{2n} = (-1)^0 \frac{0}{1!} = 0.$$

**Exercise 2.** First observe that the function

$$\begin{aligned} (1, +\infty) &\rightarrow \mathbb{R} \\ x &\mapsto \frac{1}{x \ln(x)} \end{aligned}$$

is decreasing (and piecewise continuous). Hence, by the integral comparison test, for  $N \geq 3$ ,

$$\int_2^{N+1} \frac{dx}{x \ln(x)} \leq \sum_{n=2}^N \frac{1}{n \ln(n)} = \frac{1}{2 \ln(2)} + \sum_{n=3}^N \frac{1}{n \ln(n)} \leq \frac{1}{2 \ln(2)} + \int_2^N \frac{dx}{x \ln(x)}.$$

Since

$$\int \frac{dx}{x \ln(x)} = \ln(\ln(x)) + C,$$

we obtain:

$$\forall N \geq 3, \ln(\ln(N+1)) - \ln(\ln 2) \leq S_N \leq \frac{1}{2 \ln(2)} + \ln(\ln N) - \ln(\ln 2).$$

In particular, by the Squeeze Theorem, we conclude that

$$\lim_{N \rightarrow +\infty} S_N = +\infty,$$

hence the series (S) diverges. Now, for  $N \geq 3$ ,

$$\frac{\ln(\ln(N+1)) - \ln(\ln 2)}{\ln(\ln N)} \leq \frac{S_N}{\ln(\ln N)} \leq 1 + \frac{\frac{1}{2 \ln(2)} - \ln(\ln 2)}{\ln(\ln N)},$$

and since  $\ln(\ln(N+1)) \underset{N \rightarrow +\infty}{\sim} \ln(\ln N)$  (see below) we conclude, by the Squeeze Theorem, that

$$\lim_{N \rightarrow +\infty} \frac{S_N}{\ln(\ln N)} = 1,$$

i.e.,

$$S_N \underset{N \rightarrow +\infty}{\sim} \ln(\ln N).$$

To show that  $\ln(\ln(N+1)) \underset{N \rightarrow +\infty}{\sim} \ln(\ln N)$ , we may use the Mean Value Theorem: for  $N \geq 3$ , there exists  $c_N \in (N, N+1)$  such that

$$\ln(\ln(N+1)) = \ln(\ln N) + \frac{1}{c_N \ln c_N},$$

hence

$$\frac{\ln(\ln(N+1))}{\ln(\ln N)} = 1 + \frac{1}{c_N \ln c_N \ln(\ln N)} \underset{N \rightarrow +\infty}{\rightarrow} 1.$$

**Exercise 3.**

1. Notice that

$$\begin{aligned} u_{n+1} - u_n &= -\ln(n+1) + \sum_{k=1}^{n+1} \frac{1}{k} + \ln(n) - \sum_{k=1}^n \frac{1}{k} \\ &= \ln\left(\frac{n}{n+1}\right) + \frac{1}{n+1} \\ &= \ln\left(1 - \frac{1}{n+1}\right) + \frac{1}{n+1} \\ &= \frac{1}{n+1} - \frac{1}{2(n+1)^2} + o\left(\frac{1}{n^2}\right) + \frac{1}{n+1} \\ &\underset{n \rightarrow +\infty}{\sim} \frac{1}{2n^2} < 0, \end{aligned}$$

which is the general term of a convergent series. Hence, by the equivalent test, the series  $\sum_n (u_{n+1} - u_n)$  is convergent.

2. Observe that for  $N \in \mathbb{N}$ ,

$$\sum_{n=1}^N (u_{n+1} - u_n) = u_{N+1} - u_1 = u_{N+1} - 1.$$

Now, since the series  $\sum_n (u_{n+1} - u_n)$  converges, i.e., the limit

$$\lim_{N \rightarrow +\infty} \sum_{n=1}^N (u_{n+1} - u_n) = \lim_{N \rightarrow +\infty} u_{N+1} - 1$$

exists in  $\mathbb{R}$ , we conclude that  $\lim_{N \rightarrow +\infty} u_{N+1}$  exists in  $\mathbb{R}$ , and hence that the sequence  $(u_n)_{n \geq 1}$  converges.

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3. For  $N \in \mathbb{N}$  with  $N \geq 2$ :

$$S_N = \sum_{n=2}^N \frac{1}{n} - \ln \left( \frac{n}{n-1} \right) = \sum_{n=2}^N \frac{1}{n} - (\ln(n) - \ln(n-1)) = \left( \sum_{n=2}^N \frac{1}{n} \right) - \ln(N) = u_N - 1 \xrightarrow{N \rightarrow +\infty} \gamma - 1$$

hence we conclude that the series (S) converges and that

$$\sum_{n=2}^{+\infty} \left( \frac{1}{n} - \ln \left( \frac{n}{n-1} \right) \right) = \gamma - 1.$$

**Exercise 4.** Define the sequence  $(u_n)_{n \in \mathbb{N}}$  as

$$u_n = \frac{(-1)^n}{1 + \sqrt{n}}.$$

Clearly, the sequence  $(u_n)_{n \in \mathbb{N}}$  is an alternating sequence, and for  $n \in \mathbb{N}$  one has

$$|u_{n+1}| < |u_n|.$$

Moreover,

$$\lim_{n \rightarrow +\infty} u_n = 0,$$

hence, by the Alternating Series Test, the series  $\sum_n u_n$  is convergent. We also know that

$$\forall N \geq 0, |R_N| = \left| \sum_{n=N+1}^{+\infty} u_n \right| \leq |u_{N+1}| = \frac{1}{1 + \sqrt{N+1}}.$$

Hence, a sufficient condition for  $S_N$  to be an approximation of the sum of the series (S) with error less than  $10^{-5}$  is

$$\frac{1}{1 + \sqrt{N+1}} < 10^{-5} \iff 1 + \sqrt{N+1} > 10^5 \iff N > (10^5 - 1)^2 - 1.$$

Hence, a sufficient condition is  $N \geq 10^{10}$ .

**Exercise 5.**

1. Let  $a, b \in \mathbb{R}_+^*$ . The function

$$t \mapsto \frac{e^{-at} - e^{-bt}}{t}$$

is continuous on  $\mathbb{R}_+^*$ , hence the integral  $I(a, b)$  is improper at  $0^+$  and at  $+\infty$ . Now,

$$\frac{e^{-at} - e^{-bt}}{t} = \frac{1}{t} \left( (1 - at + o(t)) - (1 - bt + o(t)) \right) \xrightarrow{t \rightarrow 0} -a + b + o(1) \xrightarrow{t \rightarrow 0} -a + b \in \mathbb{R},$$

hence the integral  $I(a, b)$  is falsely improper at  $0^+$ . Moreover,

$$\forall t \in [1, +\infty), \left| \frac{e^{-at} - e^{-bt}}{t} \right| \leq e^{-at} + e^{-bt},$$

and since  $a > 0$  and  $b > 0$ , we know that the improper integrals

$$\int_1^{+\infty} e^{-at} dt \quad \text{and} \quad \int_1^{+\infty} e^{-bt} dt$$

converge hence, by the comparison test, the improper integral  $I(a, b)$  is absolutely convergent at  $+\infty$ . Hence the improper integral  $I(a, b)$  converges.

2. Let  $A > 0$ . We use Theorem 3 with  $J = \mathbb{R}_+^*$ ,  $J = [A, +\infty)$  and the function  $u$  defined by:

$$u : \mathbb{R}_+^* \times [A, +\infty) \longrightarrow \mathbb{R} \quad (t, x) \longmapsto \frac{e^{-t} - e^{-xt}}{t}.$$

• For all  $(t, x) \in \mathbb{R}_+^* \times [A, +\infty)$  the partial derivative  $\partial_2 u(t, x)$  clearly exists and in fact:

$$\partial_2 u(t, x) = e^{-xt}.$$

• Clearly, for all  $x \in [A, +\infty)$ , the functions

$$\mathbb{R}_+^* \longrightarrow \mathbb{R} \quad t \longmapsto \frac{e^{-t} - e^{-xt}}{t} \quad \text{and} \quad \mathbb{R}_+^* \longrightarrow \mathbb{R} \quad t \longmapsto \partial_2 u(t, x) = e^{-xt}$$

are continuous.

• Define the function  $g$  as

$$g : \mathbb{R}_+^* \longrightarrow \mathbb{R} \quad t \longmapsto e^{-At}.$$

Clearly,  $g$  is continuous and

$$\forall (t, x) \in \mathbb{R}_+^* \times [A, +\infty), |\partial_2 u(t, x)| = e^{-xt} \leq e^{-At}.$$

• Moreover, the improper integral

$$\int_0^{+\infty} e^{-At} dt$$

converges (since  $A > 0$ ). We also know (from Question 1) that for all  $x \in \mathbb{R}_+^*$ , the improper integral

$$\int_0^{+\infty} u(t, x) dt$$

converges.

Hence we conclude that  $\varphi$  is differentiable on  $[A, +\infty)$  and that

$$\forall x \in [A, +\infty), \varphi'(x) = \int_0^{+\infty} e^{-xt} dt.$$

Now, since this is true for all  $A > 0$ , we conclude that  $\varphi$  is differentiable on

$$\bigcup_{A \in \mathbb{R}_+^*} [A, +\infty) = \mathbb{R}_+^*$$

and that

$$\forall x \in \mathbb{R}_+^*, \varphi'(x) = \int_0^{+\infty} e^{-xt} dt.$$

Let  $X > 0$  and  $x \in \mathbb{R}_+^*$ . Then:

$$\int_0^X e^{-xt} dt = \left[ \frac{e^{-xt}}{-x} \right]_{t=0}^{t=X} = -\frac{e^{-xX}}{x} + \frac{1}{x} \xrightarrow{X \rightarrow +\infty} \frac{1}{x}.$$

Hence

$$\forall x \in \mathbb{R}_+^*, \varphi'(x) = \frac{1}{x}.$$

3. From the previous question we conclude that there exists  $C \in \mathbb{R}$  such that

$$\forall x \in \mathbb{R}_+^*, \varphi(x) = \ln(x) + C.$$

Now, since

$$\varphi(1) = \int_0^{+\infty} \frac{e^{-t} - e^{-t}}{t} dt = \int_0^{+\infty} 0 dt = 0,$$

we conclude that  $C = 0$ , hence

$$\varphi : \mathbb{R}_+^* \longrightarrow \mathbb{R} \quad x \longmapsto \ln(x).$$

4. Let  $a, b \in \mathbb{R}_+^*$  and  $X, Y \in \mathbb{R}_+^*$  such that  $X < Y$ . Using the substitution  $s = at$ :

$$\int_X^Y \frac{e^{-at} - e^{-bt}}{t} dt = \int_{aX}^{aY} \frac{e^{-s} - e^{-bs/a}}{s/a} \frac{ds}{a} = \int_{aX}^{aY} \frac{e^{-s} - e^{-bs/a}}{s} ds = \varphi(b/a) - \ln(b/a).$$

Hence

$$\forall a, b \in \mathbb{R}_+^*, I(a, b) = \int_0^{+\infty} \frac{e^{-at} - e^{-bt}}{t} dt = \ln \left( \frac{b}{a} \right).$$