INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

# SCAN 2 — Solution of Math Test #4

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**Exercise 1.**

1. The matrix of $q$ in the standard basis of $E$ is:

$$[q]_{\text{std}} = \begin{pmatrix} 1 & 1/2 & 1 & 1 \\ 1/2 & 5 & -3/2 & 5/2 \\ 1 & -3/2 & 3 & 1 \\ 1 & 5/2 & 1 & 3 \end{pmatrix}.$$

2. Let $(x, y, z, t) \in E$. Then:

$$(x,y,z,t) \in \mathbb{R}^4 \iff \begin{cases} x+y+z+t=0 \\ x-y+z-t=0 \end{cases} \iff \begin{cases} x+\ \ y+z+\ \ t=0 \\ \quad -2y \qquad -2t=0 \end{cases} \iff \begin{cases} x=-z \\ y=-t \\ z=\ \ z \\ t=\ \ t. \end{cases}$$

hence a basis of $F$ is given by:

$$\mathscr{C} = \big((-1,0,1,0),(0,-1,0,1)\big).$$

In the sequel we'll use the following notation:

$$u_1 = (-1,0,1,0), \quad u_2 = (0,-1,0,1).$$

3.
- We first show that $\mathscr{B}$ is a family of vectors of $F^{\perp}$: Recall that a vector $u \in E$ belongs to $F^{\perp_\varphi}$ if and only if $u \perp_\varphi u_1$ and $u \perp_\varphi u_2$. Now:

$$\varphi\big((1,1,1,-1),u_1\big) = \begin{pmatrix}1 & 1 & 1 & -1\end{pmatrix}[q]_{\text{std}}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix} = \begin{pmatrix}1 & 1 & 1 & -1\end{pmatrix}\begin{pmatrix}0\\-2\\2\\0\end{pmatrix} = 0$$

$$\varphi\big((1,1,1,-1),u_2\big) = \begin{pmatrix}1 & 1 & 1 & -1\end{pmatrix}[q]_{\text{std}}\begin{pmatrix}0\\-1\\0\\1\end{pmatrix} = \begin{pmatrix}1 & 1 & 1 & -1\end{pmatrix}\begin{pmatrix}1/2\\-5/2\\5/2\\1/2\end{pmatrix} = 0$$

hence $(1,1,1,-1) \in F^{\perp_\varphi}$

$$\varphi\big((1,0,0,-1),u_1\big) = \begin{pmatrix}1 & 0 & 0 & -1\end{pmatrix}[q]_{\text{std}}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}0\\-2\\2\\0\end{pmatrix} = 0$$

$$\varphi\big((1,0,0,-1),u_2\big) = \begin{pmatrix}1 & 0 & 0 & -1\end{pmatrix}[q]_{\text{std}}\begin{pmatrix}0\\-1\\0\\1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}1/2\\-5/2\\5/2\\1/2\end{pmatrix} = 0$$

hence $(1,0,0,-1) \in F^{\perp_\varphi}$.

- We now show that $\mathscr{B}$ is orthogonal with respect to $\varphi$:

$$\varphi\big((1,1,1,-1),(1,0,0,-1)\big) = \begin{pmatrix}1 & 1 & 1 & -1\end{pmatrix}[q]_{\text{std}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} = \begin{pmatrix}1 & 1 & 1 & -1\end{pmatrix}\begin{pmatrix}0\\-2\\0\\-2\end{pmatrix} = 0.$$

- $\mathscr{B}$ is an orthogonal family and doesn't contain the nil vector, hence $\mathscr{B}$ is and independent family of vectors of $F^{\perp_\varphi}$.

- We know that $E=F\oplus F^{\perp_\varphi}$ and that $\dim E=4$ and $\dim F=2$, hence $\dim F^{\perp_\varphi}=2$. Hence, since $\mathscr{B}$ consists of two independent vectors of $F^{\perp_\varphi}$, $\mathscr{B}$ is a basis of $F^{\perp_\varphi}$.

4. We know that:

$$m=\min_{(x,y,z,t)\in F}\big\|(x-1,y-1,z-1,t-1)\big\|_\varphi^2=\big\|(1,1,1,1)-p_F\big(1,1,1,1)\big\|_\varphi^2$$

where $p_F : E\longrightarrow E$ denotes the orthogonal projection (with respect to $\varphi$) onto $F$. If we denote by $p' : E\longrightarrow E$ the orthogonal projection (with respect to $\varphi$) onto $F^{\perp_\varphi}$ we have:

$$p'(1,1,1,1)=(1,1,1,1)-p_F(1,1,1,1)\in F^{\perp_\varphi},$$

hence

$$m=\big\|p'(1,1,1,1)\big\|_\varphi^2.$$

Set: $v_1=(1,1,1,-1)$ and $v_2=(1,0,0,-1)$. Then, since $\mathscr{B}=(v_1,v_2)$ is an orthogonal basis of $F^{\perp_\varphi}$,

$$p'(1,1,1,1)=\frac{\varphi\big((1,1,1,1),v_1\big)}{\varphi(v_1,v_1)}v_1+\frac{\varphi\big((1,1,1,1),v_2\big)}{\varphi(v_2,v_2)}v_2,$$

and by the Pythagorean Theorem,

$$m=\big\|p'(1,1,1,1)\big\|_\varphi^2=\left\|\frac{\varphi\big((1,1,1,1),v_1\big)}{\varphi(v_1,v_1)}v_1\right\|_\varphi^2+\left\|\frac{\varphi\big((1,1,1,1),v_2\big)}{\varphi(v_2,v_2)}v_2\right\|_\varphi^2=\frac{\varphi\big((1,1,1,1),v_1\big)^2}{\varphi(v_1,v_1)}+\frac{\varphi\big((1,1,1,1),v_2\big)^2}{\varphi(v_2,v_2)}.$$

Now,

$$\varphi\big((1,1,1,1),v_1\big)=\begin{pmatrix}1&1&1&1\end{pmatrix}[q]_{\text{std}}\begin{pmatrix}1\\1\\1\\-1\end{pmatrix}=\begin{pmatrix}1&1&1&1\end{pmatrix}\begin{pmatrix}3/2\\3/2\\3/2\\3/2\end{pmatrix}=6$$

$$\varphi\big(v_1,v_1\big)=\begin{pmatrix}1&1&1&-1\end{pmatrix}[q]_{\text{std}}\begin{pmatrix}1\\1\\1\\-1\end{pmatrix}=\begin{pmatrix}1&1&1&-1\end{pmatrix}\begin{pmatrix}3/2\\3/2\\3/2\\3/2\end{pmatrix}=3$$

$$\varphi\big((1,1,1,1),v_2\big)=\begin{pmatrix}1&1&1&1\end{pmatrix}[q]_{\text{std}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}=\begin{pmatrix}1&1&1&1\end{pmatrix}\begin{pmatrix}0\\-2\\0\\-2\end{pmatrix}=-4$$

$$\varphi\big(v_2,v_2\big)=\begin{pmatrix}1&0&0&-1\end{pmatrix}[q]_{\text{std}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}=\begin{pmatrix}1&0&0&-1\end{pmatrix}\begin{pmatrix}0\\-2\\0\\-2\end{pmatrix}=2.$$

Hence

$$m=\frac{36}{3}+\frac{16}{2}=20.$$

**Exercise 2.** Let $y$ be a power series of radius $R>0$, say

$$y(x)=\sum_{n=0}^{+\infty}a_nx^n.$$

Then, for all $x\in(-R,R)$,

$$x^2y(x)=\sum_{n=0}^{+\infty}a_nx^{n+2}=\sum_{n=2}^{+\infty}a_{n-2}x^n,$$

$$y'(x)=\sum_{n=0}^{+\infty}na_nx^{n-1},\qquad\text{hence}\qquad xy'(x)=\sum_{n=0}^{+\infty}na_nx^n,$$

$$y''(x) = \sum_{n=0}^{+\infty} n(n-1)a_n x^{n-2}, \qquad \text{hence} \qquad x^2 y''(x) = \sum_{n=0}^{+\infty} n(n-1)a_n x^n = \sum_{n=2}^{+\infty} n(n-1)a_n x^n.$$

Hence, for all $x \in (-R, R)$,

$$x^2 y''(x) + xy'(x) + x^2 y(x) = a_1 x + \sum_{n=2}^{+\infty} \big(n(n-1)a_n + na_n + a_{n-2}\big)x^n = a_1 x + \sum_{n=2}^{+\infty} \big(n^2 a_n + a_{n-2}\big)x^n.$$

Then, the following equivalences are true:

$$\begin{aligned}
y \text{ is a solution of } (*) \text{ on } (-R,R) &\iff a_0 = 1,\ a_1 = 0,\ \forall x \in (-R,R),\ \sum_{n=2}^{+\infty} \big(n^2 a_n + a_{n-2}\big)x^n = 0 \\
&\iff \begin{cases} a_0 = 1 \\ a_1 = 0 \\ \forall n \geq 2,\ n^2 a_n + a_{n-2} = 0 \end{cases} \qquad \textit{(Identity Theorem)} \\
&\iff \begin{cases} a_0 = 1 \\ a_1 = 0 \\ \forall n \geq 0,\ a_{n+2} = -\dfrac{a_n}{(n+2)^2}. \end{cases}
\end{aligned}$$

By a simple induction, we see that the sequence $(a_n)_{n\in\mathbb{N}}$ satisfies this last equivalence if and only if:

$$\begin{cases} \forall p \in \mathbb{N},\ a_{2p+1} = 0 \\ a_0 = 1 \\ \forall p \in \mathbb{N}^*,\ a_{2p} = \dfrac{(-1)^p}{\big(2 \times 4 \times \cdots \times (2p)\big)^2} \end{cases}$$

Now, for all $p \in \mathbb{N}^*$,

$$\big(2 \times 4 \times \cdots \times (2p)\big) = 2^p \big(1 \times 2 \times \cdots \times p\big)^2 = 4^p (p!)^2.$$

Hence, the power series $y$ is a solution of $(*)$ on $(-R, R)$ if and only if:

$$\forall p \in \mathbb{N}, \quad \begin{cases} a_{2p+1} = 0 \\ a_{2p} = \dfrac{(-1)^p}{4^p (p!)^2}, \end{cases}$$

i.e., the unique solution $y$ of $(*)$ that possesses a power series expansion is:

$$y(x) = \sum_{p=0}^{+\infty} \frac{(-1)^p}{4^p (p!)^2} x^{2p}.$$

We now compute the radius of convergence $R$ of this power series using the ratio test: let $x \in \mathbb{R}^*$. Then:

$$\left| \frac{(-1)^{p+1} x^{2p+2}}{4^{p+1}\big((p+1)!\big)^2} \frac{4^p (p!)^2}{(-1)^p x^{2p}} \right| = \frac{x^2}{4(p+1)^2} \underset{p\to+\infty}{\longrightarrow} 0 < 1.$$

Hence the power series converges on $\mathbb{R}$ hence $R = +\infty$.
*Remark.* The function $y$ we obtained is known as the Bessel function of order 0 of the first kind. It appears naturally when solving the wave equation on a disk.

**Exercise 3.**

1. Let $f, g \in E$. Clearly, $\varphi(f, g) = \varphi(g, f)$. We now show that $\varphi$ is symmetric with respect to the first argument: let $f, g, h \in E$ and $\lambda \in \mathbb{R}$. Then:

$$\varphi(f + \lambda g, h) = \int_0^1 \big(f(t) + \lambda g(t)\big)h(t)\,\mathrm{d}t + \int_0^1 \big(f'(t) + \lambda g'(t)\big)h'(t)\,\mathrm{d}t$$

$$= \int_0^1 f(t)h(t)\,\mathrm{d}t + \lambda \int_0^1 g(t)h(t)\,\mathrm{d}t + \int_0^1 f'(t)h'(t)\,\mathrm{d}t + \lambda \int_0^1 g'(t)h'(t)\,\mathrm{d}t = \varphi(f,h) + \lambda\varphi(g,h).$$

Hence $\varphi$ is a symmetric bilinear form on $E$. We now show that $\varphi$ is positive semi-definite: let $f \in E$, then, clearly,

$$\varphi(f,f) = \int_0^1 f(t)^2\,\mathrm{d}t + \int_0^1 \big(f'(t)\big)^2\,\mathrm{d}t \geq 0.$$

We now show that $\varphi$ is positive definite: let $f \in E$ such that $\varphi(f,f) = 0$. Then:

$$\int_0^1 f(t)^2\,\mathrm{d}t + \int_0^1 f'(t)^2\,\mathrm{d}t = 0.$$

Since both integrals are non-negative, we must have:

$$\int_0^1 f(t)^2\,\mathrm{d}t = 0$$

and since $f^2$ is non-negative *and continuous* on $[0,1]$, we must have $f = 0$ on $[0,1]$, i.e., $f = 0_E$.

2. Let $f \in E$. We remember (hopefully!) from the lectures about differential equations that:

$$f'' - f = 0 \iff \exists A, B \in \mathbb{R},\ \forall x \in [0,1],\ f(x) = A\mathrm{e}^x + B\mathrm{e}^{-x}.$$

Define the following functions:

$$e_+ : \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \mathrm{e}^x, \end{array} \qquad\qquad e_- : \begin{array}{ccc} [0,1] & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \mathrm{e}^{-x}. \end{array}$$

Clearly the vectors $e_+$ and $e_-$ are independent and hence they form a basis of $V$:

$$V = \operatorname{Span}\{e_+, e_-\}, \qquad \dim V = 2 < +\infty.$$

3. Let $f \in V$ and $g \in E$. Then, by integration by parts:

$$\begin{aligned}
\varphi(f,g) &= \int_0^1 f(t)g(t)\,\mathrm{d}t + \int_0^1 f'(t)g'(t)\,\mathrm{d}t \\
&= \int_0^1 f(t)g(t)\,\mathrm{d}t + \big[f'(t)g(t)\big]_{t=0}^{t=1} - \int_0^1 f''(t)g'(t)\,\mathrm{d}t \\
&= \int_0^1 \big(f(t) - f''(t)\big)g(t)\,\mathrm{d}t + f'(1)g(1) - f'(0)g(0) \\
&= f'(1)g(1) - f'(0)g(0).
\end{aligned}$$

Now,

$$g \in V^{\perp_\varphi} \iff \varphi(e_+,g) = \varphi(e_-,g) = 0 \iff \begin{cases} \mathrm{e}g(1) - g(0) = 0 \\ -\mathrm{e}^{-1}g(1) + g(0) = 0 \end{cases} \iff g(0) = g(1) = 0.$$

Hence $V^{\perp_\varphi} = \{g \in E \mid g(0) = g(1) = 0\}$.

4. Let $f \in E$. Observe that $e_+$ and $e_-$ are orthogonal with respect to $\varphi$:

$$\varphi(e_+, e_-) = \int_0^1 \mathrm{e}^t\mathrm{e}^{-t}\,\mathrm{d}t + \int_0^1 \mathrm{e}^t(-\mathrm{e}^{-t})\,\mathrm{d}t = 0$$

hence:

$$p(f) = \frac{\varphi(f,e_+)}{\varphi(e_+,e_+)}e_+ + \frac{\varphi(f,e_-)}{\varphi(e_-,e_-)}e_-.$$

Now,

$$\begin{aligned}
\varphi(f,e_+) &= f(1)\mathrm{e} - f(0) \qquad \textit{(by question 3 since } e_+ \in V\textit{)}, \\
\varphi(f,e_-) &= f(1)\mathrm{e}^{-1} - f(0) \qquad \textit{(by question 3 since } e_- \in V\textit{)}, \\
\varphi(e_+,e_+) &= \mathrm{e}^2 - 1, \\
\varphi(e_-,e_-) &= 1 - \mathrm{e}^{-2}.
\end{aligned}$$

Hence

$$p(f) = \frac{f(1)\mathrm{e} - f(0)}{\mathrm{e}^2 - 1}e_+ + \frac{f(1)\mathrm{e}^{-1} - f(0)}{1 - \mathrm{e}^{-2}}e_-.$$