INSA LYON

SCAN 2 — Math Test #4 — 1h30

April 24, 2014

No documents, no calculators, no cell phones or electronic devices allowed.

Exercise 1. Let $E=\mathbb{R}^4$ and define the following positive definite quadratic form[1] on E:

$$q:\begin{array}{ccc} E & \longrightarrow & \mathbb{R} \\ (x,y,z,t) & \longmapsto & 2x^2+3y^2+4z^2+t^2+2xt-4yz \end{array}$$

The associated inner product is denoted by φ and the associated norm is denoted by $\|\cdot\|_\varphi$. We define the following family $\mathscr{B}$ of vectors of E:

$$\mathscr{B}=\big((1,1,1,0),(0,6,5,8)\big),$$

and we define the following subspace F of E:

$$F=\big\{(x,y,z,t)\in E \;\big|\; x+y+z+t=0,\ x-y+z-t=0\big\}.$$

1. Write the matrix of q in the standard basis of E.
2. Determine a basis $\mathscr{C}$ of F.
3. Show that the family $\mathscr{B}$ is an orthogonal (with respect to φ) basis of $F^\perp$.
4. Compute the value of

$$m=\min_{(x,y,z,t)\in F}\big\|(x-1,y-1,z-1,t-1)\big\|_\varphi^2.$$

Exercise 2. Find all the solutions (if any!) of the following IVP:

$$(*)\qquad \begin{cases} x^2y''(x)+xy'(x)+x^2y(x)=0 \\ y(0)=1 \\ y'(0)=0 \end{cases}$$

that admit a power series expansion. If there are any solutions, determine explicitly the coefficients as well as the radius of convergence of the power series you obtained.

Exercise 3. We denote by $E=C^1\big([0,1],\mathbb{R}\big)$ the real vector space that consists of all real valued functions of class C^1 on $[0,1]$. We define the mapping

$$\varphi:\begin{array}{ccc} E\times E & \longrightarrow & \mathbb{R} \\ (f,g) & \longmapsto & \displaystyle\int_0^1 f(t)g(t)\,\mathrm{d}t+\int_0^1 f'(t)g'(t)\,\mathrm{d}t. \end{array}$$

1. Show that φ is an inner product on E.
2. Define
$$V=\big\{f\in E \;\big|\; f \text{ is of class } C^2 \text{ on } [0,1] \text{ and } f''=f\big\}.$$
Show that V is a finite dimensional subspace of E and give a basis of V.
3. Define
$$W=\big\{f\in E \;\big|\; f(0)=f(1)=0\big\}.$$
Show that W is a subspace of E and that $V\perp_\varphi W$.
4. Show that V and W are complementary subspaces in E (i.e., $E=V\oplus W$).
5. We denote by p the orthogonal projection of E onto V. For $f\in E$, explicit $p(f)$.
6. Let $(\alpha,\beta)\in\mathbb{R}^2$. Define
$$E_{\alpha,\beta}=\big\{f\in E \;\big|\; f(0)=\alpha,\ f(0)=\beta\big\}.$$
Compute
$$\inf_{f\in E_{\alpha,\beta}}\int_0^1\big(f(t)^2+f'(t)^2\big)\,\mathrm{d}t.$$

[1] you don't have to prove that q is an positive definite quadratic form on E. We have checked that for you.