SCAN 2 — Math Test #4— 1h30 *May 3, 2012*

No document, no calculator, no mobile-phone allowed.

Exercise 1. Let $\lambda > 0$. We consider the differential equation

$$(E_\lambda) \qquad xy''(x) + 2y'(x) + \lambda xy(x) = 0.$$

The goal is to find the solutions of (E_λ) that possess a power series expansion.

1. Let $f(x) = \sum_{n=0}^{+\infty} a_n x^n$ be a power series with radius R that is a solution of (E_λ). Find the relations between the a_n's and deduce that
$$\forall p \in \mathbb{N}, \begin{cases} a_{2p} = \dfrac{(-\lambda)^p}{(2p+1)!} a_0 \\ a_{2p+1} = 0. \end{cases}$$

2. Deduce the value of R and express the solutions of (E_λ) on $(-R, R)$ that possess a power series expansion, in terms of usual functions.

Exercise 2. Questions 2 and 3 of this exercise are independent from each other.

1. Let E be a real vector space and let $\langle \cdot | \cdot \rangle$ be an inner product on E, and $\| \cdot \|$ the associated norm. Say why for all $u, v \in E \setminus \{0\}$, $\dfrac{\langle u|v \rangle}{\|u\| \|v\|} \in [-1, 1]$.

   Definition. The number $\theta = \arccos \left(\dfrac{\langle u|v \rangle}{\|u\| \|v\|} \right)$ is called the *non-oriented angle between u and v with respect to $\langle \cdot | \cdot \rangle$.*

2. In this question only we consider $E = \mathbb{R}^4$. We consider the vectors $u = (0, 1, 0, 1)$ and $v = (1, 1, 1, 1)$ of E.
   a) Compute the non-oriented angle between u and v with respect to the standard dot product of $E = \mathbb{R}^4$.
   b) We now consider the following quadratic form on E:
   $$q(x, y, z, t) = x^2 + 2xy + 2y^2 + z^2 + t^2,$$
   and we denote by φ the associated polar form.
      i) Show that φ is an inner product on E.
      ii) Write the matrix $[q]_{\text{std}}$ of the quadratic form q in the standard basis of E.
      iii) Compute the cosine of the non-oriented angle between u and v with respect to φ.
      iv) Compute the orthogonal projection (with respect to φ) of the vector $w = (1, 0, 0, 0)$ on the subspace $F = \text{Span}\{u, v\}$.

3. In this question only we consider the vector space $E = \mathbb{R}_3[X]$, that consists of polynomials of degree at most 3 with real coefficients, together with the symmetric bilinear form φ on E given by
   $$\varphi(P, Q) = P(0)Q(0) + \int_{-1}^{1} P'(t)Q'(t) \, \mathrm{d}t.$$
   We recall that the standard basis of E is $\mathscr{C} = \left(1, X, X^2, X^3\right)$. We also consider the following vectors of E:
   $$S_1 = X^3, \qquad S_2 = 5X^3 - 9X, \qquad S_3 = 3X^2 - 8, \qquad S_4 = X^2 + 1.$$
   a) Show that φ is an inner product on E and explicit the matrix $A = [\varphi]_{\mathscr{C}}$. Is the basis $\mathscr{C}$ an orthogonal basis of E?
   b) What is the cosine of the non-oriented angle between X and X^3 with respect to φ?

c) You are given that $\mathscr{B} = (S_1, S_2, S_3, S_4)$ is a basis of E. Explicit the change of bases matrix $P = [\mathscr{B}]_{\mathscr{C}}$ from the basis $\mathscr{C}$ to the basis $\mathscr{B}$. Use the change of basis formula to show that

$$[\varphi]_{\mathscr{B}} = \begin{pmatrix} 18/5 & 0 & 0 & 0 \\ 0 & 72 & 0 & 0 \\ 0 & 0 & 88 & 0 \\ 0 & 0 & 0 & 11/3 \end{pmatrix}.$$

What can you deduce about $\mathscr{B}$?

d) We now consider the vector $Q = 1$ of E. Compute the orthogonal projection of the vector Q on the subspace $F = \mathrm{Span}\{S_1, S_2, S_3\}$.

Exercise 3. Let E be a real vector space and let $\langle \cdot | \cdot \rangle$ be an inner product on E. We denote by $\| \cdot \|$ the associated Euclidean norm. Let f be an endomorphism of E such that

$$\forall u \in E, \ \big\| f(u) \big\| = \|u\|.$$

1. Prove that for all $u, v \in E$, $\big\langle f(u) \big| f(v) \big\rangle = \big\langle u \big| v \big\rangle$. Hint: Compute $\big\| f(u+v) \big\|^2$.

2. If E is finite-dimensional and $\mathscr{B}$ is an orthonormal basis of E, show that $A = [f]_{\mathscr{B}}$ is an orthogonal matrix (i.e., ${}^tAA = \mathrm{Id}$).

Exercise 4. Let $a > 0$. Consider the function f, periodic on $\mathbb{R}$ of period $2a$, defined on $(-a, a]$ by

$$f(x) = \begin{cases} -1 & \text{if } x \in (-a, 0) \\ 0 & \text{if } x = 0 \text{ or } x = a \\ 1 & \text{if } x \in (0, a). \end{cases}$$

1. Plot the graph of f on $[-2a, 4a]$.

2. Write the Fourier series $S(f)$ of f.

3. Do we have $S(f) = f$? Justify your answer.

4. Is the series $S(f)$ uniformly convergent on $\mathbb{R}$?