INSA LYON

SCAN 2 — Test #4 — 1h30 *May 19, 2011*

No document, no calculator, no mobile-phone allowed.

Exercise 1. We consider the mapping φ from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by

$$\varphi(x,y) = \big(x\mathrm{e}^{-y}, x^2+y\big).$$

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a function of class C^2 on $\mathbb{R}^2$ and set $f = F \circ \varphi$. Find all points (x_0, y_0) in $\mathbb{R}^2$ such that φ is a local C^2-diffeomorphism in a neighborhood of (x_0, y_0), and compute $\dfrac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$ in terms of the partial derivatives of F.

Exercise 2. Find the position of the local extreme values on $\mathbb{R}^2$ of the function f defined by

$$f(x,y) = 12xy - 2xy^2 - x^3,$$

and specify it they correspond to local minimum values or local maximum values.

Exercise 3. We consider the quadratic form q on $\mathbb{R}^3$ defined by

$$q(x,y,z) = 4x^2 + y^2 + z^2 - 6yz.$$

We denote by $\mathscr{B}$ the standard basis of $\mathbb{R}^3$.

1. Write the matrix $B = [q]_{\mathscr{B}}$ of q in the basis $\mathscr{B}$.
2. What is the signature of q? Find a basis $\mathscr{C}$ of $\mathbb{R}^3$ that is orthonormal with respect to the standard dot product of $\mathbb{R}^3$, such that the matrix $C = [q]_{\mathscr{C}}$ of q in the basis $\mathscr{C}$ is diagonal. You will explicit the change of basis matrix $P = [\mathscr{C}]_{\mathscr{B}}$ from the basis $\mathscr{B}$ to the basis $\mathscr{C}$ and give a relation between B, C and P.
3. We consider the subspace F of $\mathbb{R}^3$ given by

$$F = \big\{(x,y,z) \in \mathbb{R}^3 \;\big|\; y+z=0\big\}.$$

a) Find two vectors u and v of $\mathscr{C}$ that form a basis of F, and prove that for all $\lambda, \mu \in \mathbb{R}$,

$$q(\lambda u + \mu v) = 4\lambda^2 + 4\mu^2.$$

b) We consider the surface (Σ) of $\mathbb{R}^3$ defined by

$$(\Sigma) = \big\{(x,y,z) \in \mathbb{R}^3 \;\big|\; q(x,y,z) = 1\big\}.$$

What is the nature of the curve $(\Sigma) \cap F$? Plot $(\Sigma) \cap F$ in $(O; u, v)$.

Please turn over

Exercise 4. We consider the sequence $(b_n)_{n\in\mathbb{N}^*}$ given by

$$b_n = \begin{cases} 0 & \text{if } n = 2 \\ \dfrac{8}{\pi} \cdot \dfrac{(-1)^n - 1}{n\big(n^2-4\big)} & \text{if } n \neq 2. \end{cases}$$

Consider the function f defined on $\mathbb{R}$ by

$$f(x) = \sum_{n=1}^{+\infty} b_n \sin(nx).$$

1. Show that f is continuous on $\mathbb{R}$, periodic of period 2π and odd.
2. Show that for all $x \in [0,\pi]$, $f(x) = 1 - \cos(2x)$.
3. Sketch the graph of f on $[-2\pi, 2\pi]$.
4. For $n \in \mathbb{N}^*$ we consider the following differential equation:

$$(\mathrm{E}_n) \qquad y'(x) + y(x) = b_n \sin(nx).$$

a) For $n \in \mathbb{N}^*$, check that a particular solution on $\mathbb{R}$ of the differential equation (E_n) is

$$y_n(x) = A_n \cos(nx) + B_n \sin(nx)$$

where

$$A_n = -\frac{nb_n}{n^2+1}, \qquad B_n = \frac{b_n}{n^2+1}.$$

b) Prove that for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}^*$,

$$\big|y_n(x)\big| \leq 2|b_n|, \qquad \big|y_n'(x)\big| \leq 2|b_n|.$$

c) We consider the series of functions s_P defined by

$$s_P(x) = \sum_{n=1}^{+\infty} y_n(x) = \sum_{n=1}^{+\infty} A_n \cos(nx) + B_n \sin(nx).$$

Prove that s_P is of class C^1 on $\mathbb{R}$ and that it can be differentiated term by term on $\mathbb{R}$.

d) Deduce that s_P is a particular solution of the differential equation

$$(*) \qquad y' + y = f$$

on $\mathbb{R}$, and deduce the general form of the solutions of Equation $(*)$ on $\mathbb{R}$.