SCAN 2 — Test #4 — 1h30 *May 7, 2010*

No document, no calculator, no mobile-phone allowed.

Exercise 1.

1. Give the definition of a scalar product.
2. In the vector space $\mathbb{R}^2$ we define the following quadratic form:

$$q(x,y) = x^2 + y^2 + xy.$$

   a) Give the matrix associated with the quadratic form q in the canonical basis of $\mathbb{R}^2$.

   b) We consider the following basis of $\mathbb{R}^2$:

$$\mathcal{B} = \big((1,1),(1,-1)\big).$$

   Give the matrix associated with the quadratic form q in the basis $\mathcal{B}$.

   c) Compute $\varphi\big((1,1),(1,-1)\big)$, where φ is the polar form associated with the quadratic form q.

Exercise 2.

1. Give the radius of convergence of the power series

$$\sum_{n=1}^{+\infty} n x^n$$

   and compute its sum.
2. We consider the following rational function:

$$\varphi(x) = \frac{x}{(1-x)^2\big(1+x^2\big)}.$$

   Give the power series expansion of φ, and specify the domain of validity.
3. We want to solve the following system of sequences, for $n \in \mathbb{N}$:

$$\begin{cases} a_{n+1} + b_n = n \\ a_n - b_{n+1} = 1 \\ a_0 \in \mathbb{R},\ b_0 = -1. \end{cases}$$

   We consider the following power series:

$$f(x) = \sum_{n=0}^{+\infty} a_n x^n.$$

   a) Show that

$$f(x) = \frac{a_0 + x(1-2a_0) + a_0 x^2}{\big(1+x^2\big)(1-x)^2}.$$

   b) In the case $a_0 = 0$, deduce an expression of a_n and b_n in terms of n.

Exercise 3. Let f be the function defined by:

$$f(x) = 1 + \pi - \frac{8}{\pi} \sum_{n=0}^{+\infty} \frac{\cos(2n+1)x}{(2n+1)^2}.$$

1. a) Show that f is continuous on $\mathbb{R}$.
   b) Show, using Fourier series, that for all $x \in [0, \pi]$,

$$f(x) = 2x + 1.$$

   c) Sketch the graph of the function f on the interval $[-2\pi, 2\pi]$.
   d) Give the value of the sum of the series:

$$\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2}.$$

2. Let $n \in \mathbb{N}$ and $\alpha_n \in \mathbb{R}$. We consider the following linear differential equation

$$(E_n) \qquad y'' + 4y' + y = \alpha_n \cos nt.$$

   a) Show that the function y_n defined by

$$\forall t \in \mathbb{R},\ y_n(t) = A_n \cos nt + B_n \sin nt$$

   is a solution of Equation (E_n) if

$$A_n = \frac{\alpha_n \left(1 - n^2\right)}{\left(1 - n^2\right)^2 + 16n^2}, \qquad B_n = \frac{4n\alpha_n}{\left(1 - n^2\right)^2 + 16n^2}.$$

   b) Give a simple equivalent of A_n and B_n when $n \to +\infty$.
   c) Let $(\alpha_n)_{n \geq 0}$ be a sequence of real numbers such that the series $\sum |\alpha_n|$ converges. Deduce from the previous question a particular solution of the following linear differential equation:

$$y'' + 4y' + y = \sum_{n=0}^{+\infty} \alpha_n \cos nt.$$

   d) Deduce the solutions of the following linear differential equation:

$$y'' + 4y' + y = f.$$