INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Math Test #3— 1h30

March 21, 2013

No documents, no calculators, no mobile-phones allowed.

Exercise 1.

1. Give the power series expansion of $f(t) = \ln(1+t)$, and specify the radius of convergence of this power series.

2. We assume that
$$y(x) = \sum_{n=0}^{+\infty} a_n x^n$$
is a power series with radius of convergence $R > 0$. Determine a necessary and sufficient condition on the a_n's for y to be a solution of the following differential equation
$$\text{(E)} \qquad x^2 y''(x) + 2xy'(x) - 2y(x) = \ln\big(1+x^2\big)$$
on $(-R, R)$, and determine R.

3. We consider the function g defined by a power series as
$$g(x) = \sum_{p=1}^{+\infty} \frac{(-1)^{p+1}}{p(2p+2)(2p-1)} x^{2p}.$$

   a) Determine the radius of convergence R_2 of g.

   b) Give an expression of g on $(-R_2, R_2)$. *Hint: use a partial fraction decomposition.*

   c) Show that g is defined and continuous on $[-R_2, R_2]$.

   d) Compute $g(R_2)$.

Exercise 2. We consider the series of functions f defined by $f(x) = \displaystyle\sum_{n=1}^{+\infty} (-1)^{n-1} \frac{n}{x^2+n^2}$.

1. Show that f is well-defined and of class C^1 on $\mathbb{R}$.

2. a) Show that for all $x \in [0,1]$ the numerical sequence $\left(\dfrac{n}{x^2+n^2}\right)_{n\in\mathbb{N}^*}$ is decreasing.

   b) Show that $\displaystyle\int_0^1 f(x)\,\mathrm{d}x = \sum_{n=1}^{+\infty} (-1)^{n-1} \arctan\left(\frac{1}{n}\right)$.

3. a) Show that for all $n \in \mathbb{N}^*$, and for all $x \in \mathbb{R}$,
$$\int_0^{+\infty} \cos(xt) \mathrm{e}^{-nt}\,\mathrm{d}t = \frac{n}{x^2+n^2}.$$

   b) Show that for all $N \in \mathbb{N}^*$, and for all $x \in \mathbb{R}$ we can write
$$\sum_{p=1}^{N} (-1)^{p-1} \frac{p}{x^2+p^2} = \int_0^{+\infty} \frac{\cos(xt)}{\mathrm{e}^t+1}\,\mathrm{d}t + R_N(x)$$
   where $\big|R_N(x)\big| \leq \dfrac{1}{N}$.

   c) Deduce that for all $x \in \mathbb{R}$,
$$f(x) = \int_0^{+\infty} \frac{\cos(xt)}{\mathrm{e}^t+1}\,\mathrm{d}t.$$

Exercise 3. We consider the vector space $E = \mathbb{R}^2$ with the standard basis $\mathscr{B} = \big((1,0),(0,1)\big)$ and the basis $\mathscr{C} = \big((2,1),(-1,1)\big)$. We also consider the quadratic form q on E, the matrix of which in the basis $\mathscr{B}$ is

$$[q]_{\mathscr{B}} = A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

1. Briefly justify that $\mathscr{C}$ is a basis of E.
2. For $u = (x, y) \in E$, explicit $q(u)$.
3. Let φ be the polar form of q. For $u = (x, y), v = (x', y') \in E$, explicit $\varphi(u, v)$.
4. Explicit the matrix $A' = [q]_{\mathscr{C}}$ of q in the basis $\mathscr{C}$ using the following two methods:
   a) by a direct method.
   b) with the change of basis formula.