SCAN 2 — Math Test #3— 1h30

March 15, 2012

No document, no calculator, no mobile-phone allowed.

Exercise 1. We consider the sequence of functions $(f_n)_{n\in\mathbb{N}}$ on $\mathbb{R}_+$ defined by

$$f_n(x) = \mathrm{e}^{-nx}\cos(nx).$$

1. Find the pointwise limit of the sequence of functions $(f_n)_n$ on $\mathbb{R}_+$.
2. a) Does $(f_n)_n$ converge uniformly on $\mathbb{R}_+$?
   b) Does $(f_n)_n$ converge uniformly on $(0, +\infty)$?
   c) Does $(f_n)_n$ converge uniformly on $[a, b]$ with $0 < a < b$?

Exercise 2. We recall the following results, and you may use them without any justification:

- $\forall t > 0$, $\arctan t + \arctan \frac{1}{t} = \frac{\pi}{2}$;
- $\forall t \in \mathbb{R}$, $|\arctan t| \leq |t|$;
- $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

We consider the series of functions

$$f(x) = \sum_{n=1}^{+\infty} \frac{\arctan(nx)}{n^2}.$$

1. Show that f is a continuous, bounded and odd function on $\mathbb{R}$.
2. What are the variations of f on $\mathbb{R}$?
3. Prove that f is of class C^1 on $\mathbb{R}_+^*$ and give an expression of f' on $\mathbb{R}_+^*$.
4. For $x \in \mathbb{R}_+^*$, we set $\varphi(x) = \frac{\pi^3}{12} - f(x)$.
   a) Show that there exists $K \in \mathbb{R}$ such that $\forall x > 0$, $|\varphi(x)| \leq \frac{K}{x}$.
   b) Deduce that f has a finite limit at $+\infty$ and determine the value of this limit.
5. Use a comparison with an integral to prove that for all $x > 0$, $f'(x) \geq \frac{1}{2}\ln\left(\frac{1+x^2}{x^2}\right)$. Deduce the value of $\lim\limits_{x\to 0^+} f'(x)$.
6. Sketch the graph of f on $\mathbb{R}$ (you will show all the elements that have been obtained in the previous questions on your graph).

Exercise 3. We consider the power series $g(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

1. Prove that g is well-defined and continuous on $[-1, 1]$.
2. a) Prove that
$$\forall x \in (-1, 1), \; \arctan(x) = g(x).$$
   b) Is the previous equality still valid on $[-1, 1]$? Justify your answer.
3. We now consider the power series
$$f(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)(2n+2)} x^{2n+1}$$
and the differential equation

(E) $$xy'(x) + y(x) = \arctan(x).$$

   a) Determine the radius of convergence R of the power series f.

   b) Prove that f is a solution of (E) on $(-R, R)$.

   c) Give an expression of f on $(-R, R)$ in terms of usual functions. *Hint: You may either use a partial fraction decomposition of* $\frac{1}{(2n+1)(2n+2)}$ *or solve the differential equation* (E) *or use another method of your choice. We recall that for all* $x \in (-1, 1)$, $\ln(1 + x) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^n}{n}$.