INSA LYON

SCAN 2 — Test #3 — 1h30 *March 17, 2011*

No document, no calculator, no mobile-phone allowed.

Exercise 1.

1. Give the radius of convergence R_1 of the power series $\displaystyle\sum_{n=1}^{+\infty} \frac{x^n}{n}$ and give its sum on $(-R_1, R_1)$.
2. Give the radius of convergence R_2 of the power series $\displaystyle\sum_{n=0}^{+\infty} \frac{n}{n+1} x^n$ and give its sum on $(-R_2, R_2)$.
3. Deduce the value of the sum of the series

$$\sum_{n=0}^{+\infty} 2^{n+1} \mathrm{e}^{-nx} \frac{n}{n+1}$$

and specify the domain of validity.

Exercise 2. We consider the sequence of functions $(f_n)_{n\in\mathbb{N}}$ defined on $[0, +\infty)$ by

$$f_n(x) = \frac{x^n}{1 + x^{2n}}.$$

1. Compute the pointwise limit of the sequence of functions $(f_n)_{n\in\mathbb{N}}$ on $[0, +\infty)$.
2. Does the series of functions (f_n) converge uniformly on $[0, +\infty)$? If $a > 0$, does the series of functions (f_n) converge uniformly on $[0, a]$? on $[a, +\infty)$? (Discuss according to the value of a).

Exercise 3. The goal of this exercise is to compute the sum of the series

$$S = \sum_{n=1}^{+\infty} \frac{\sin n}{n}.$$

1. Let $x \in (-1, 1)$. Prove that the series

$$\sum_{n=0}^{+\infty} x^n \mathrm{e}^{inx}$$

converges, and compute the value of its sum.

2. Deduce the values of the sums

$$\sum_{n=0}^{+\infty} x^n \cos nx, \qquad \sum_{n=0}^{+\infty} x^n \sin nx.$$

Prove that the corresponding series of functions converge uniformly on $[-a, a]$ for all $a \in (0, 1)$.

3. For $n \in \mathbb{N}^*$ and $x \in \mathbb{R}$ we set

$$u_n(x) = \frac{x^n \sin nx}{n},$$
$$f_n(x) = \sum_{k=1}^{n} u_k(x),$$

and we consider the series of functions

$$f = \sum_{n=1}^{+\infty} u_n.$$

a) Prove that the series of functions f converges on $(-1, 1)$.
b) Prove that f is of class C^1 on $(-1, 1)$ and express f' in terms of usual functions on $(-1, 1)$.
c) A computer algebra system yields:

> $\int -\frac{x^2 - \cos(x)x - \sin(x)}{x^2 - 2\cos(x)x + 1} \, dx$

$$-\frac{x}{2} - \text{ArcTan}\left[\frac{(x+1)\text{Tan}\left[\frac{x}{2}\right]}{x-1}\right]$$

Deduce that for all $x \in (-1, 1)$,

$$f(x) = -\frac{x}{2} - \arctan\left(\frac{(x+1)\tan(x/2)}{x-1}\right).$$

4. a) Let $N \in \mathbb{N}$, $N \geq 2$ and $x \in [0, 1]$. Prove the following equalities:

$$f_N(x) = \sum_{n=1}^{N} \frac{x^n}{n} \sin(nx) = \sum_{n=1}^{N} \frac{x^n}{n} \big(B_n(x) - B_{n-1}(x)\big) = \frac{x^N}{N} B_N(x) + \sum_{n=1}^{N-1} \left(\frac{x^n}{n} - \frac{x^{n+1}}{n+1}\right) B_n(x),$$

where

$$B_n(x) = \sum_{k=0}^{n} \sin(kx).$$

b) Show that the series of functions of x

$$\sum_{n=1}^{+\infty} \left(\frac{x^n}{n} - \frac{x^{n+1}}{n+1}\right)$$

converges normally on $[0, 1]$.
c) It can be shown, and you may use this result without any justification, that for all $x \in (0, 1]$,

$$B_n(x) = \frac{\sin\left(\frac{n}{2}x\right)\sin\left(\frac{n+1}{2}x\right)}{\sin\frac{x}{2}}.$$

Show that the series of functions of x

$$\sum_{n=1}^{+\infty} \left(\frac{x^n}{n} - \frac{x^{n+1}}{n+1}\right) B_n(x)$$

converges normally on $[\pi/4, 1]$.
d) Prove that the sequence of functions $(h_n)_{n \geq 1}$ defined on $[\pi/4, 1]$ by

$$h_n(x) = \frac{x^n}{n} B_n(x)$$

converges uniformly to the zero function on $[\pi/4, 1]$.
e) Deduce that f is continuous on $[\pi/4, 1]$ and deduce that

$$S = \sum_{n=1}^{+\infty} \frac{\sin n}{n} = \frac{\pi - 1}{2}.$$