INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Solution of Math Test #1

November 6, 2014

Romaric Pujol, romaric.pujol@insa-lyon.fr

Exercise 1.

1. Let $A \in \mathbb{R}$. The function $t \mapsto e^{-t}$ is continuous on $[A, +\infty)$ hence the improper integral $\int_A^{+\infty} e^{-t}\,dt$ is improper at $+\infty$. Let $B \in \mathbb{R}$ with $B > A$. One has:
$$\int_A^B e^{-t}\,dt = e^{-A} - e^{-B}.$$
Since $\lim_{B\to+\infty} e^{-B} = 0$ we conclude that the improper integral $\int_A^B e^{-t}\,dt$ converges and
$$\int_A^{+\infty} e^{-t}\,dt = e^{-A}.$$

2. Let $x \in \mathbb{R}_+^*$. The function $t \mapsto \frac{e^{-t}}{t}$ is continuous on $[x, +\infty)$ hence the improper integral is improper at $+\infty$. Now, for $t \in [x, +\infty)$ one has
$$0 \le \frac{e^{-t}}{t} \le \frac{1}{x} e^{-t}.$$
Since the improper integral
$$\int_x^{+\infty} e^{-t}\,dt$$
converges (Question 1) we conclude, by the Comparison Theorem, that the improper integral $\int_x^{+\infty} \frac{e^{-t}}{t}\,dt$ is convergent.
The case $x = 0$:
$$\frac{e^{-t}}{t} \underset{t\to 0^+}{\sim} \frac{1}{t} > 0.$$
Now, the improper integral
$$\int_0^1 \frac{dt}{t}$$
diverges at 0 (Riemann at 0 with $\alpha = 1$), hence the improper integral $\int_0^{+\infty} \frac{e^{-t}}{t}\,dt$ diverges at 0, hence diverges.

3. For $x \in \mathbb{R}_+^*$ one has
$$F(x) - F(1) = \int_x^{+\infty} \frac{e^{-t}}{t}\,dt - \int_1^{+\infty} \frac{e^{t}}{t}\,dt = \int_x^1 \frac{e^{-t}}{t}\,dt \qquad \textit{(since both integrals converge)}$$
hence
$$F(x) = F(1) - \int_1^x \frac{e^{-t}}{t}\,dt.$$
Since the function $g : t \mapsto -\frac{e^{-t}}{t}$ is continuous on $\mathbb{R}_+^*$, we conclude that that F is an antiderivative of g on $\mathbb{R}_+^*$, hence F is of class C^1 on $\mathbb{R}_+^*$ and $F' = g$, i.e.,
$$\forall x \in \mathbb{R}_+^*, \quad F'(x) = g(x) = -\frac{e^{-x}}{x}.$$

4. a) Let $x \in \mathbb{R}_+^*$. Then, for $t \in [x, +\infty)$,
$$0 \le \frac{x}{t} e^{-t} \le e^{-t}$$
hence, integrating with respect to t from x to $+\infty$ (as all the improper integrals are convergent) yields:
$$0 \le xF(x) \le \int_x^{+\infty} e^{-t}\,dt.$$
b) For $x \in \mathbb{R}_+^*$,
$$\int_x^{+\infty} e^{-t} = e^{-x}$$
(Question 1) hence, by the Squeeze Theorem, $\lim_{x\to+\infty} xF(x) = 0$.

5. a) Let $x \in (0,1]$. Then, for $t \in [x, 1]$,
$$0 \le \frac{e^{-t}}{t} \le \frac{1}{t}$$
hence, integrating with respect to t from x to 1 (since $x \le 1$) yields
$$0 \le \int_x^1 \frac{e^{-t}}{t}\,dt = F(x) - F(1) \le \int_x^1 \frac{dt}{t} = -\ln x,$$
hence the result.
b) We conclude that for $x \in (0,1]$ one has
$$0 \le xF(x) \le xF(1) - x\ln x$$
and since $\lim_{x\to 0^+} x\ln x = 0$ we conclude, by the Squeeze Theorem, that
$$\lim_{x\to 0^+} xF(x) = 0.$$

6. Let $A, B \in \mathbb{R}_+^*$ with $A < B$. Since F is of class C^1 on $[A, B]$, we obtain:
$$\int_A^B F(x)\,dx = \big[xF(x)\big]_{x=A}^{x=B} - \int_A^B xF'(x)\,dx = BF(B) - AF(A) - \int_A^B e^{-x}\,dx = BF(B) - AF(A) + e^{-A} - e^{-B}.$$
Since $\lim_{x\to+\infty} xF(x) = 0$, taking the limit as B approaches $+\infty$ yields the convergence of the following improper integral:
$$\int_A^{+\infty} F(x)\,dx = -AF(A) + e^{-A}$$
and since $\lim_{x\to 0^+} xF(x) = 0$, taking the limit as A approaches 0^+ yields
$$\int_0^{+\infty} F(x)\,dx = 1.$$

Exercise 2.

1.
- First observe that N only takes values in $\mathbb{R}_+$ (since we're integrating a non-negative function from 0 to 1 with $0 < 1$).
- Let $f \in E$ such that $N(f) = 0$, i.e.,
$$\int_0^1 t\big|f(t)\big|\,dt = 0.$$
Since the function $t \mapsto t\big|f(t)\big|$ is continuous on $[0,1]$, we must have:
$$\forall t \in [0,1], \quad t\big|f(t)\big| = 0,$$
hence
$$\forall t \in (0,1], \quad f(t) = 0.$$
Since f is continuous, we conclude that $f = 0_E$.
- Let $f, g \in E$. Then, for $t \in [0,1]$,
$$t\big|f(t) + g(t)\big| \le t\big|f(t)\big| + t\big|g(t)\big|$$
and integrating this inequality with respect to t from 0 to 1 (with $0 < 1$) yields:
$$N(f+g) = \int_0^1 t\big|f(t)+g(t)\big| \le \int_0^1 t\big|f(t)\big| + \int_0^1 t\big|g(t)\big|\,dt = N(f) + N(g).$$

- Let $f \in E$ and $\lambda \in \mathbb{R}$. Then:

$$N(\lambda f) = \int_0^1 t\big|\lambda f(t)\big|\,\mathrm{d}t = |\lambda| \int_0^1 t\big|f(t)\big|\,\mathrm{d}t = |\lambda| N(f).$$

Hence N is a norm on E.

2. Let $f \in E$. One has:

$$\forall t \in [0,1], \quad t\big|f(t)\big| \leq \big|f(t)\big|$$

hence, integrating with respect to t from 0 to 1 (with $0 < 1$) yields

$$N(f) = \int_0^1 t\big|f(t)\big|\,\mathrm{d}t \leq \int_0^1 \big|f(t)\big|\,\mathrm{d}t = \|f\|_1.$$

3. Observe that for all $n \in \mathbb{N}^*$, f_n only takes non-negative values, hence $|f_n| = f_n$.

   a) Let $n \in \mathbb{N}^*$. Then:

$$N(f_n) = \int_0^1 t\big|f_n(t)\big|\,\mathrm{d}t = \int_0^{1/n} tn(1-nt)\,\mathrm{d}t = \int_0^{1/n} \big(nt - n^2t^2\big)\,\mathrm{d}t = \frac{1}{2n} - \frac{1}{3n} = \frac{1}{6n}.$$

   and

$$\|f_n\|_1 = \int_0^1 \big|f_n(t)\big|\,\mathrm{d}t = \int_0^{1/n} n(1-nt)\,\mathrm{d}t = n\left(\frac{1}{n} - \frac{1}{2n}\right) = \frac{1}{2}.$$

   b) Since $\lim_{n\to+\infty} N(f_n) = 0$, we conclude that the sequence $(f_n)_{n\in\mathbb{N}^*}$ converges to 0_E for the norm N.

   c) Since $\lim_{n\to+\infty} \|f_n - 0_E\|_1 = 1/2 \neq 0$, we conclude that the sequence $(f_n)_{n\in\mathbb{N}^*}$ doesn't converge to 0_E for the norm $\|\cdot\|_1$.

   d) The norms N and $\|\cdot\|_1$ are not equivalent, because the convergence of the sequence $(f_n)_{n\in\mathbb{N}^*}$ is not the same for the norm N and the norm $\|\cdot\|_1$.

4. A necessary and sufficient condition is that

$$\min_{x\in[0,1]} g(x) > 0.$$

In this case, setting $\alpha = \min_{x\in[0,1]} g(x) > 0$ and $\beta = \max_{x\in[0,1]} g(x)$ yields:

$$\forall f \in E, \quad \alpha\|f\|_1 \leq N(f) \leq \beta\|f\|_1.$$

Exercise 3. Let $(x_0, y_0) \in \mathbb{R}^2$, and let $(h,k) \in \mathbb{R}^2$. Then:

$$\begin{aligned} f(x_0+h, y_0+k) &= \big(x_0^2 + 2x_0h + h^2 + y_0^2 + 2y_0k + k^2, x_0y_0 + x_0k + y_0h + hk\big) \\ &= \big(x_0^2 + y_0^2, x_0y_0\big) + \big(2x_0h + 2y_0k, x_0k + y_0h\big) + \big(h^2+k^2, hk\big) \\ &= f(x_0,y_0) + \varphi(h,k) + \big(h^2+k^2, hk\big), \end{aligned}$$

where φ is the *linear mapping* defined by

$$\begin{aligned} \varphi : \quad \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (h,k) &\longmapsto (2x_0h + 2y_0k, x_0k + y_0h). \end{aligned}$$

Observe that since we're in a finite dimensional vector space, the mapping φ is continuous, whatever norm we choose.

In order to prove that f is differentiable at (x_0, y_0) it only remains to prove that

$$\lim_{(h,k)\to(0,0)} \frac{\big\|(h^2+k^2, hk)\big\|}{\big\|(h,k)\big\|} = 0$$

where $\|\cdot\|$ is any norm (we're in a finite dimensional vector space, hence all norms are equivalent). We choose the 1-norm. Then, for $(h,k) \in \mathbb{R}^2$:

$$\begin{aligned} 0 \leq \big\|(h^2+k^2, hk)\big\|_1 &= \big|h^2+k^2\big| + \big|hk\big| \\ &\leq |h|^2 + |k|^2 + |h||k| \\ &\leq 3\big\|(h,k)\big\|_1^2, \end{aligned}$$

where we have used the inequalities

$$|h| \leq |h| + |k| = \big\|(h,k)\big\|_1 \qquad \text{and} \qquad |k| \leq |h| + |k| = \big\|(h,k)\big\|_1.$$

Hence, for all $(h,k) \in \mathbb{R}^2 \setminus \big\{(0,0)\big\}$ one has

$$0 \leq \frac{\big\|(h^2+k^2, hk)\big\|_1}{\big\|(h,k)\big\|_1} \leq 3\big\|(h,k)\big\|_1$$

hence, by the Squeeze Theorem,

$$\lim_{\|(h,k)\|_1\to 0} \frac{\big\|(h^2+k^2, hk)\big\|_1}{\big\|(h,k)\big\|_1} = 0.$$

We conclude that f is differentiable at (x_0, y_0) and that its differential at (x_0, y_0) is φ, namely:

$$\begin{aligned} D_{(x_0,y_0)}f : \quad \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (h,k) &\longmapsto (2x_0h + 2y_0k, x_0k + y_0h). \end{aligned}$$