Physics Semester Exam
Monday 26 January 2015

Preliminary remarks :
Duration : 3 hrs

- Problem I and II are totally independent of each other.
- Marking scheme (approximate) : PART 1: 12 points, PART 2: 8 points.
- Any numerical result given without units will not be taken in consideration.
- Authorised Calculator: non programmable, "college" model .

1 Problem I: Vehicle detection loop

Inducting loop vehicle detectors are devices whose goal is to trigger traffic lights or to regulate congestion by detecting the presence or the passing by of a vehicle. They are made of inducting loops buried under the road at a depth h (Figure 1). When a car arrives above the inducting loop, the influence of the magnetic field on the metallic frame of the car causes Eddy current to arise in the metallic part of the vehicle. These currents modify in turn the magnetic flux going through the inducting loop. The objective of this problem is to study the working principle of these loops.

Throughout the subject a time-varying current $i(t)$ is considered to flow through the loop. We will consider that the Quasi Static Assumption is obeyed and we will neglect the phenomenon of propagation of the electromagnetic quantities involved in the description of the device.

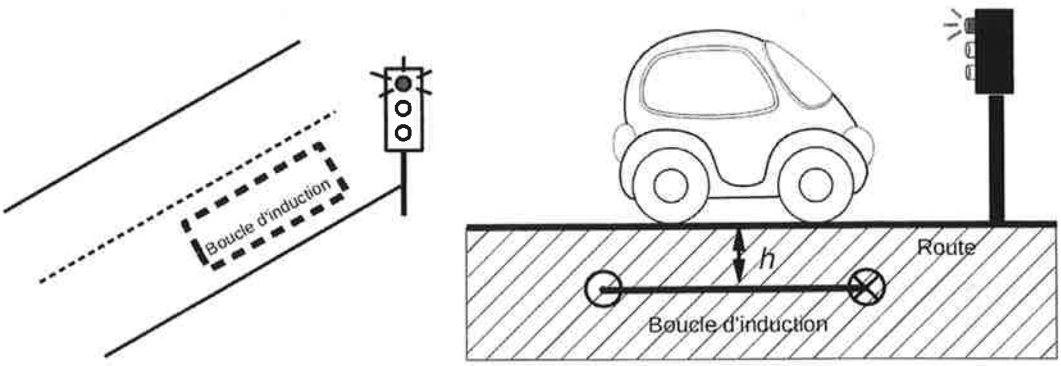


Figure 1: Inducting loop buried under the road for the detection of cars

1.1 Study of the inducting loop

An inducting loop is in fact a rectangular frame constituted of metallic cables having a circular cross-section of radius e =0.7 mm. The dimensions of the frame are respectively $a = 1$ m and $b = 2$ m.

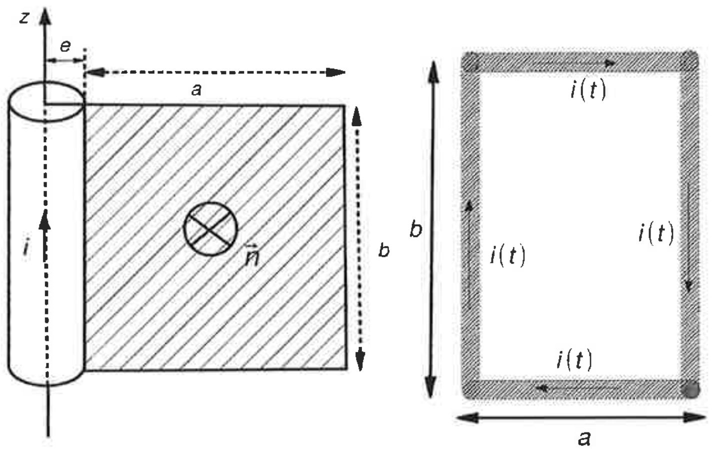


Figure 2: Left: Infinite wire considered in the calculation of the contribution of one single branch to the total flux. Right: Global geometry of the inducting loop.

In order to sub-divide the problem, we will first consider only ***one branch*** *of the four which constitute the inducting loop, and we will consider it equivalent to a* ***solid infinite cylinder*** *of radius $e \ll a, b$, through which a current density $\vec{j} = j(t)\vec{u_z}$ is flowing. This current density is assumed to be uniform and associated with a total current $i(t)$. Initially, we will consider a cylindrical coordinate system $\vec{u_r}$, $\vec{u_\theta}$, $\vec{u_z}$. $\vec{j}$ is oriented along the axis zz' of the cylinder.*

Question 1.1 : Express $i(t)$ as a function of $j(t)$ and e.

Question 1.2 : Analyse in a very detailed manner the symmetries and invariances of the current distribution. Give the general expression of the magnetic field fulfilling the symmetries and invariances of the problem.

Question 1.3 : Calculate the magnetic field inside and outside the cable as a function of $i(t)$.

Question 1.4 : Deduce the flux going through the oriented rectangular surface of length a and b indicated on the figure 2, as a function of $i(t)$, a, b, e and μ_0. The normal to use for this calculation is indicated on the Figure 2 (left).

We consider now all 4 branches of the detection loop and the Cartesian coordinate system indicated on the Figure 3 is used. Edge effects are neglected, which is equivalent to saying that the 4 branches are considered as infinite wires.

Question 1.5 : Calculate the total flux Φ_S due to the 4 branches inside the loop itself.

Question 1.6 : Deduce the inductance of the entire loop as a function of μ_0, a, b and e.

Question 1.7 : Using solely symmetry considerations, indicate the direction or at least the non-zero components of the total magnetic field created by the 4 branches at the points A (axis zz'), B (plane of the loop) and C (medial plane of one side of the loop)

The loop manufacturer specifies that the total inductance of the loop should lie between 70 and 500 μH. To achieve this value, the loop will be made of N ***turns*** *to increase its inductance.*

Question 1.8 : How does the flux generated by N loops conducting a current i vary as a function of N ? What is the lowest number of turns that have to be used to reach the specification? What is the highest?

The influence of a vehicle located above the inducting loop causes a variation in this inductance. In order to detect the variation of inductance, we connect the inducting loop in series with a capacitance C_0.

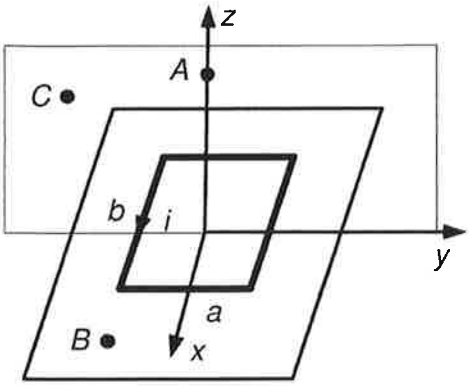


Figure 3: Topography of the field at different particular points

Question 1.9 : (bonus) If the inducting loop is modelled by a perfect inductance L in series with a resistance r, propose a practical method to detect the change in inductance (the basic principle is enough, no need to go into technical details).

1.2 Study of induced currents

*The inducting loop is considered equivalent to **only two infinite wires** oriented parallel to the xx' axis and through which a current $i(t)$ flows. In practice, this can be done if one side of the loop is much longer than the other, in other words if $b \gg a$. The loop is buried at a depth h = 0.5 m below the road, whose dielectric and magnetic properties will be considered as those of vacuum.*

*To simplify the description, the metallic frame of the vehicle is considered equivalent to a **conductor occupying the whole half-space** $z > 0$. We will consider the magnetic and dielectric properties of the conductor to also be those of **vacuum** and we will neglect throughout the whole problem the height of the tyres.*

*The magnetic field generated by the inducting loop is time-varying with a frequency f large enough for us to assume that when a vehicle places itself above the inducting loop, the bottom part of the metallic frame is submitted to Eddy currents over a width δ **negligible** with respect to the width of the conductor. Hence all induced currents are assumed to be **superficial** and will be denoted $\vec{k_f}$.is positioned*

*The total magnetic field in the whole space $\vec{B_T}$ is the **sum** of the magnetic field generated by the inducting loop $\vec{B_S}$ and of the magnetic field generated by the Eddy currents $\vec{B_F}$. **Under the conditions mentioned above, we will assume that the total field $\vec{B_T} = \vec{B_S} + \vec{B_F}$ is nil in the metallic frame of the car**, i.e. for $z > 0$.*

We will now use the Cartesian coordinate system indicated on Figure 4.

Question 1.10 : Which physical phenomena can cause the existence of Eddy currents in the metallic frame? (Give a precise and detailed answer)

*We focus now on the case where the car has **stopped** (it is motionless) above the inducting loop. The Eddy currents that arise in the car's metallic frame themselves generate an induced magnetic field $\vec{B_F}$. We are interested in the **total** magnetic field $\vec{B_T} = \vec{B_S} + \vec{B_F}$ generated by **the whole system constituted of the <u>two</u> inducting wires and of the Eddy currents induced by the two inducting wires.***

Question 1.11 : We assume that the Eddy currents are collinear to $\vec{u_x}$. Using the boundary conditions, prove that $\vec{B}_{T(z=0^-)}$ is collinear to $\vec{u}_y$.

To model in the half plane $z < 0$ the total magnetic system made of the inducting loop and of the metallic frame of the car, and taking into account the boundary conditions (and also the symmetries) mentioned previously, we propose the following equivalent model: it consists of two infinite wires located in a plane parallel to the (xOy) plane at a distance $z = h$ from the ground (and consequently

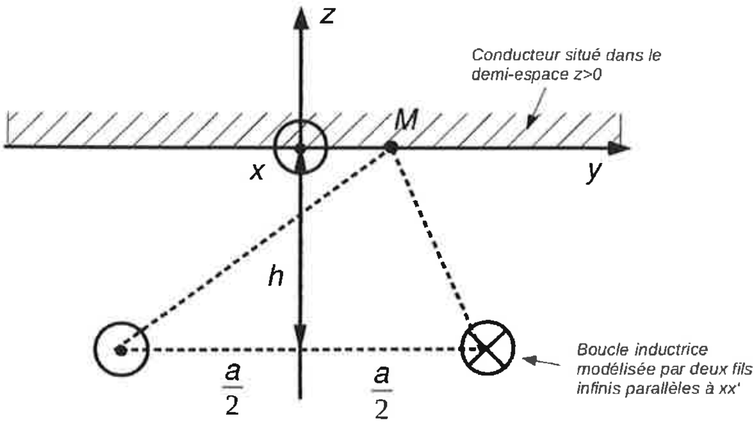


Figure 4: Sketch showing the two infinite wires representing the inducting loop and the semi-infinite conductor representing the car parked on the road.

at a distance $z = 2h$ from the wires of the inducting loop). The position of these two wires is thus symmetrical to the wires of the inducting loop with respect to the plane $z = 0$. ***The current flowing through the two wires modelling the car has the same amplitude (i) as that of the current flowing through the inducting loop.***

Question 1.12 : Complete and hand in the sketch provided in annex 1, identical to that shown on Figure 5, by adding the direction of the current flow within the wires modelling the car. Consider carefully both possible solutions and **clearly justify your choice** based on the symmetries of the system (an unjustified choice will score zero marks).

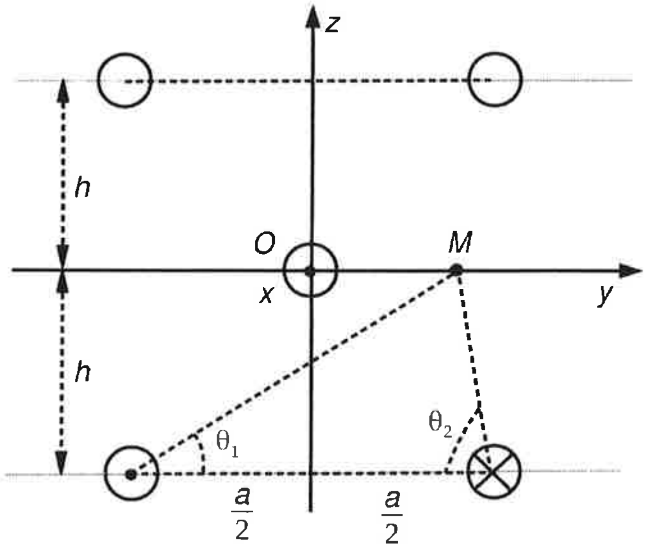


Figure 5: Equivalent circuit model of the ***complete*** *system$\vec{B_S} + \vec{B_F}$ (to be completed on the sketch in annex 1).*

We will calculate the flux sent into the inductive coil by the Eddy currents which we have modelled by two infinite wires, as discussed above.

Question 1.13 : Draw on the figure provided in the annex 2, i.e. on the copy identical to that shown on Figure 6, the contribution of each wire, located on the plane $z = +h$, to the field $\vec{B_F}$ produced at point M (belonging to the plane $z = -h$).

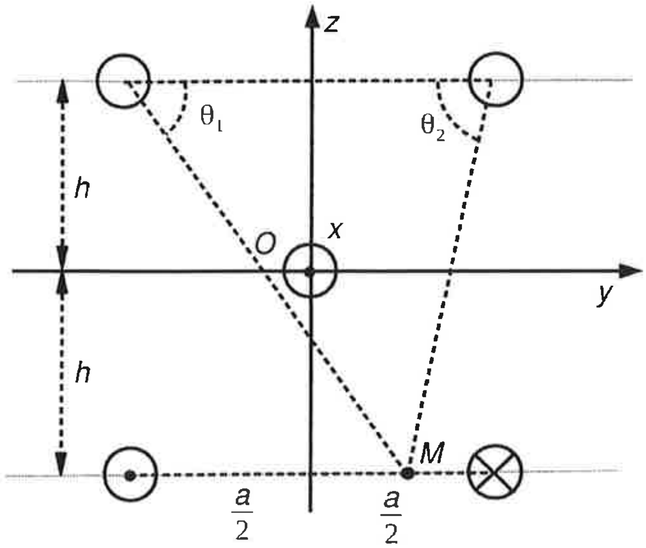


Figure 6: Sketch to be completed in annex 2 for the flux sent into the inductive loop by the vehicle parked on the road

Question 1.14 : Calculate, as a function of μ_0, $i(t)$, h, y and a, the component along $\vec{u}_z$ of the magnetic field $\vec{B}_F$ at point $M(0, y, -h)$ produced by the two infinite wires located on the plane $z = +h$.

Question 1.15 : From this, deduce that the flux sent by the stationary vehicle into the inductive loop of length b along the xx' axis can be expressed as follows (in absolute terms) :

$$|\Phi_{FS}| = \frac{\mu_0 |i(t)| b}{2\pi} ln(1 + \frac{a^2}{4h^2})$$

The normal of the inducting loop is oriented along $\vec{u}_z$.

Note:

$$\int \frac{u}{1+u^2} du = \frac{1}{2} ln(1+u^2) + constante$$

Question 1.16 : Based on this model, explain why and in what way the presence of the Eddy currents influence the total inductance of the inductive coil in the presence of a vehicle (you can answer this question without knowing the answer to the previous question)

Question 1.17 : Give the expression and numerical value of the change in inductanceΔL due to the presence of the vehicle above the inductive loop (when N the number of turns in the loop verifies $N = 1$). For the numerical application: $a = 1\ m$, $b = 2\ m$, $h = 0,2\ m$, $e = 0.7\ mm$.

Question 1.18 : We define the sensitivity of the inductive loop as $S = \frac{\Delta L}{L}$. Calculate the value of S for N=1. Bonus Question: if we increase the number of turns N (i.e. $N > 1$) in the inductive loop, does that improve the sensitivity S ?

2 Problem II: Propagation along a two-wire line

Introduction:

The network cables used to connect standard machines to the Ethernet network are typically cables comprising four pairs of twisted wires (figure 7a): each pair of wires constitute a two-wire transmission line, a conduit along which the information-carrying signal can propagate.

We will now derive a simple model to enable us to deduce ***the original characteristics of*** *the numerical signal transmitted by a computer to the network. In order to do so, we will assume that* *the four twisted pairs of wires within the cable have no affect on each other. We will therefore consider a single twisted pair of wires. Additionally, as a first approximation, we will assume that the twisting does not affect the local geometric properties of the pair of wires.*

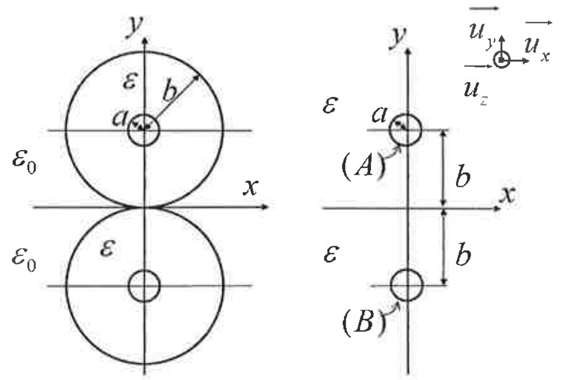
The model under study, shown on figure 7b, is therefore a single two-wire line, formed from two copper wires of diameter $2a$, of infinite length and whose axes are separated by a distance $2b$. They are separated by a dielectric which is linear, homogeneous and isotropic (LHI), of permittivity $\varepsilon = \varepsilon_r \varepsilon_0$. We assume that $a \ll b$ and that the medium surrounding the conductors is homogeneous, the conductors are therefore in effect located within an infinite LHI dielectric (figure 7c).

II.1 - Capacitance per unit length of line

The proximity of the two conductors leads to capacitive effects between them (there is electrostatic induction between one line and the other). As a first attempt to express this inter-line capacitance, we will use a very simple model of the two conductors in the line and will therefore assume that the conductors are perfect (infinite conductivity) and that ***the electrostatic induction between them is the strongest possible.***

(a) Example of two-wire lines (U/FTP standard with shielding of each pair)

(b) Sketch of the two conductors and their insulating sheath



(c) Cross-sectional view perpendicular to the axis z: on the left, the actual system; on the right, an approximation based on a uniform and homogeneous permittivity

Figure 7: Two-wire line and the associated model

Initially we will consider only one of the conductors (A), represented as a cylinder carrying a uniform surface charge density σ_1, which we assume is isolated and located within a LHI medium of permittivity ε. We denote by r_1 the distance between a point M in space and the axis of the conductor (A). To further simplify the problem, we will represent the conductor (A) as a wire charged with a linear charge densityλ_1.

Question 2.1 : What is the link between the surface charge density σ_1 and the linear charge density λ_1?

In the rest of this problem, we will work only with the linear charge density λ_1.

Question 2.2 : Establish in a detailed manner the topography of the electric field created by the linear charge density λ_1. To answer this question, use a cylindrical system of coordinates associated with the conductor (A).

Question 2.3 : Calculate, in cylindrical coordinates, the electric field $\overrightarrow{E_1}$ created by conductor (A) at a point M at distance r_1 from the central axis of conductor (A) such that $r_1 > a$. Express $\overrightarrow{E_1}$ as a function of λ_1, ε, and r_1.

We will now consider the association of (A) with a second conductor, called (B). (B) carries a linear charge density λ_2, and is supposed to be isolated and located in a uniform and homogeneous media of permittivity ε. The distance between point M and the central axis of conductor (B) is denoted r_2 .

Question 2.4 : Considering all of the above assumptions, what is the relation between the linear charge densities λ_1 and λ_2 ?

Question 2.5 : Represent on a sketch the electric fields $\overrightarrow{E_1}$ and $\overrightarrow{E_2}$ created respectively by the conductor (A) and (B) at an arbitrary point M outside both conductors. For a point on the y-axis, give the expression of the total electric field created by both conductors as a function of λ_1, ε, r_2 and r_1.

Given the approximation $a \ll b$, the initial conditions are as follows: for $r_1 = a$ and $r_2 \simeq 2b$, point M is assumed to be in contact with conductor (A) and at a potential V_A. For $r_2 = a$ and $r_1 \simeq 2b$, point M is assumed in contact with conductor (B) and at a potential V_B.

Question 2.6 : Using the circulation of the electric field along the y-axis, give the expression of the potential difference $V_A - V_B$ between the conductors (A) and (B) as a function of λ_1, ε, a and b.

Question 2.7 : Deduce that the expression of the capacity dC of a small portion of length dz of the two-wire line is $dC \simeq \frac{\pi\varepsilon}{\ln\frac{2b}{a}}dz$.

II.2 - Speed of propagation of the waves in the line

We have shown in II.1 that a length dz of the two-wire line has a capacity $dC \simeq \frac{\pi\varepsilon}{\ln\frac{2b}{a}}dz = kdz$. Additionally, from the magnetic flux created by the two wires through a surface of length dz along the two wires, it is possible to establish that a portion of length dz of the two-wire line has a self inductance $dL \simeq \frac{\mu_0}{\pi}\ln\frac{2b}{a}dz = \ell dz$ (still assuming $a \ll b$). One can also take into account, at least initially, the resistance $dR = \frac{\rho}{s}dz = rdz$ of each of the wires, with ρ the resistivity of the conductors, and $s = \pi a^2$ their cross-sectional surface area.

An electrical model of the two-wire line is given by the sketch of figure 8: one can consider each portion dz of the two-wire line as a quadripole (two-terminal-pair-network) linking the portions upstream and down stream with the circuit of figure 8.

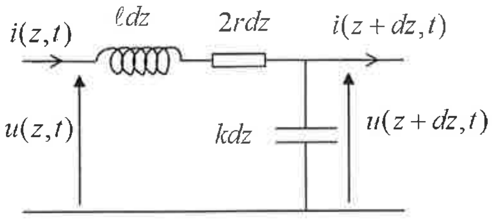


Figure 8: Model of a transmission line

In this part, we will use the approximations $f(z+dz,t) - f(z,t) \simeq \frac{\partial f(z,t)}{\partial z}dz$, and $\frac{\partial f(z+dz,t)}{\partial t} \simeq \frac{\partial f(z,t)}{\partial t}$.

Question 2.8 : From Kirchhoff's laws applied to the quadripole equivalent to the portion of the line between z and $z + dz$, and from the voltages $u(z + dz, t)$ and $u(z, t)$ between the 2 lines and the currents in each wire $i(z + dz, t)$ and $i(z, t)$ defined on figure 8, show that:

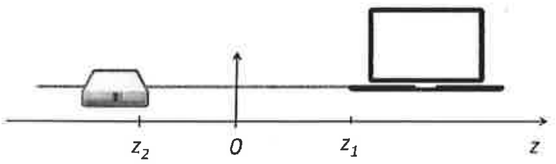
$$\frac{\partial u}{\partial z}(z,t) = -2r\,i(z,t) - \ell\frac{\partial i}{\partial t}(z,t) \quad \textit{and} \quad \frac{\partial i}{\partial z}(z,t) = -k\frac{\partial u}{\partial t}(z,t)$$

Question 2.9 : Deduce the equation of propagation verified by the voltage $u(z,t)$ along the two-wire line. In the simple case of a line with no loss ($2rdz$ neglected), give the expression of the speed of propagation of the voltage waves along the two-wire line (we will assume both wires are perfect conductors of resistivity $\rho = 0$), first as a function of k and ℓ, and then as a function of μ_0 and ε.

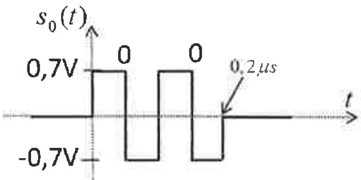
Question 2.10 : Numerical application: give the value of the speed of propagation in a transmission line where $\varepsilon_r \simeq 2,25$. Note: $\varepsilon_0 = \dfrac{10^{-9}}{36\pi}\,F.m^{-1}$ and $\mu_0 = 4\pi 10^{-7} H.m^{-1}$.

II.3- Application: signals transmitted by the line

We consider a computer connected to a first "repeater" on a network by an Ethernet cable like the one studied in II.2, with a speed of propagation $V \simeq 2.10^8 m/s$. The origin is taken to be in the middle of the cable. The computer is at the end $z_1 = +50\ m$ of the cable and the repeater at the other end, at $z_2 = -40\ m$. In the Ethernet standard, the coding of a binary number is associated with the variation of the signal every clock period (Manchester coding for example). We give here as an example the coding of a piece of information comprising two consecutive «0» bits (figure 9b) with a binary rate of 10 Mbits/s and a clock frequency of 10 MHz.



(a) Computer linked to the network



(b) Manchester coding of a signal «00»

Figure 9: Application to data transmission

Question 2.11 : The signal $s_0(t)$ is emitted at $t = 0$ s by the computer at $z_1 = +50\ m$. Find the time t_1 at which the signal begins to reach the other end of the cable $z_2 = -40$m.

Question 2.12 : Represent, as a function of z, the signal on the line at $t = t_1$.

Question 2.13 : Give the expression of $s(z, t_1)$ as a function of $s_0(t)$. Comment.