INSA | INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Math Test #2 — 1h30

December 11, 2014

No documents, no calculators, no cell phones or electronic devices allowed but you may keep your pet blobfish for moral support.

All your answers must be fully justified, unless noted otherwise.

Exercise 1. The goal of this exercise is to find all functions $f : \mathbb{R} \times \mathbb{R}_+^* \longrightarrow \mathbb{R}$ of class C^1 that satisfy the following equation:

$$\text{(E)} \qquad \forall (x,y) \in \mathbb{R} \times \mathbb{R}_+^*, \quad y\partial_1 f(x,y) - x\partial_2 f(x,y) = f(x,y).$$

1. A sufficient condition: let $h : \mathbb{R}_+^* \longrightarrow \mathbb{R}$ be a function of class C^1 and define

$$\begin{array}{rccl} f : & \mathbb{R} \times \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & h(x^2+y^2)\exp\left(\arctan\left(\dfrac{x}{y}\right)\right). \end{array}$$

   Show that f is a solution of Equation (E).

2. Let $f : \mathbb{R} \times \mathbb{R}_+^*$ be a function of class C^1 and define

$$\begin{array}{rccl} g : & \mathbb{R}_+^* \times \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) & \longrightarrow & \mathbb{R} \\ & (r,\theta) & \longmapsto & f(r\sin\theta, r\cos\theta). \end{array}$$

   Briefly check that g is well-defined and of class C^1. Compute $\partial_2 g$ and deduce the general solution (of class C^1) of Equation (E).

Exercise 2. Let $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a function of class C^2. We assume that there exists $\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}$ of class C^1 such that for all $(x,y) \in \mathbb{R}^2$ the Jacobian of φ at the point (x,y) has the form:

$$J_{(x,y)}\varphi = \begin{pmatrix} \cos\bigl(\theta(x,y)\bigr) & -\sin\bigl(\theta(x,y)\bigr) \\ \sin\bigl(\theta(x,y)\bigr) & \cos\bigl(\theta(x,y)\bigr) \end{pmatrix}$$

i.e., $J_{(x,y)}\varphi$ is a rotation matrix of angle $\theta(x,y)$. Show that there exists $\theta_0 \in \mathbb{R}$ and $(x_0, y_0) \in \mathbb{R}^2$ such that:

$$\forall (x,y) \in \mathbb{R}^2, \quad \varphi(x,y) = (x_0 + x\cos\theta_0 - y\sin\theta_0, y_0 + x\sin\theta_0 + y\cos\theta_0).$$

Hint: denote by φ_1 and φ_2 the components of φ; write the differentials of φ_1 and φ_2 (you'll get the information from the Jacobian matrix); using the fact that these differentials are exact (hence closed), deduce that θ is constant, say θ_0, and conclude.

Exercise 3. The two questions in this exercise are independent from each other.

1. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function of class C^1 and define the function g as

$$\begin{array}{rccl} g : & \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & f\big(f(x, \cos x), f(\cos x, x)\big). \end{array}$$

   For $x \in \mathbb{R}$, determine $g'(x)$ in terms of the partial derivatives of f at well-chosen points (that you will explicitly mention).

2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function of class C^2 and define the function g as

$$\begin{array}{rccl} g : & \mathbb{R}^* \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ & (x, y) & \longmapsto & f\left(\frac{y}{x}\right). \end{array}$$

   For $(x, y) \in \mathbb{R} \times \mathbb{R}^*$, compute $\partial^2_{1,2} g(x, y)$.

Exercise 4.

1. Let $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a function of class C^1 such that $\varphi(0,0) = (0,0)$ and the Jacobian matrix of φ at $(0,0)$ is

$$J_{(0,0)}\varphi = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

   Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function of class C^1, and define $g = f \circ \varphi$.

   a) Show that the directional derivatives of f and g at $(0,0)$ in the direction $(1,1)$ are equal. What can you say about the directional derivative of g at $(0,0)$ in the direction $(1,-1)$?

   b) Show that $(1,1)$ and $(1,-1)$ are eigenvectors of $D_{(0,0)}\varphi$ for eigenvalues you will determine.

2. More generally, let $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be functions of class C^1 and define $g = f \circ \varphi$. Let $x_0 \in \mathbb{R}^n$ and $\vec{u}$ be an eigenvector of $D_{x_0}\varphi$ associated with the eigenvalue λ. Give a relation between the directional derivative of g at x_0 in the direction $\vec{u}$ and the directional derivative of f at $\varphi(x_0)$ in the direction $\vec{u}$.