INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 2 — Solution of Math Test #2

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Exercise 1.

1. We compute the partial derivatives of f at a point $(x,y)\in\mathbb{R}\times\mathbb{R}_+^*$:

$$\partial_1 f(x,y)=\left(2xh'(x^2+y^2)+\frac{1}{y}\cdot\frac{1}{1+(x/y)^2}h(x^2+y^2)\right)\exp\left(\arctan\left(\frac{x}{y}\right)\right)$$
$$\partial_2 f(x,y)=\left(2yh'(x^2+y^2)-\frac{x}{y^2}\cdot\frac{1}{1+(x/y)^2}h(x^2+y^2)\right)\exp\left(\arctan\left(\frac{x}{y}\right)\right)$$

hence

$$\begin{aligned}y\partial_1 f(x,y)-x\partial_2 f(x,y)&=\left(\frac{1}{1+(x/y)^2}+\frac{x^2}{y^2}\cdot\frac{1}{1+(x/y)^2}\right)h(x^2+y^2)\exp\left(\arctan\left(\frac{x}{y}\right)\right)\\&=h(x^2+y^2)\exp\left(\arctan\left(\frac{x}{y}\right)\right)\\&=f(x,y).\end{aligned}$$

2. If $(r,\theta)\in\mathbb{R}_+^*\times\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ then $r\cos\theta\in\mathbb{R}_+^*$, hence $(r\sin\theta,r\cos\theta)$ belongs to $\mathbb{R}\times\mathbb{R}_+^*$, which is the domain of f, hence g is well-defined. The function g is of class C^1, as a composition of functions of class C^1.
Now, for $(r,\theta)\in\mathbb{R}_+^*\times\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$:

$$\partial_2 g(r,\theta)=r\cos\theta\,\partial_1 f(r\sin\theta,r\cos\theta)-r\sin\theta\,\partial_2 f(r\sin\theta,r\cos\theta).$$

- Let f be a solution of Equation (E). Then, from the previous computation we observe that we must have:

$$\partial_2 g=g.$$

Hence there exists a function $h:\mathbb{R}_+^*\longrightarrow\mathbb{R}$ of class C^1 such that

$$\forall(r,\theta)\in\mathbb{R}_+^*\times\left(-\frac{\pi}{2},\frac{\pi}{2}\right),\quad g(r,\theta)=h(r)e^\theta.$$

Now, for $(x,y)\in\mathbb{R}\times\mathbb{R}_+^*$ and $(r,\theta)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$,

$$(x,y)=(r\sin\theta,r\cos\theta)\iff r=\sqrt{x^2+y^2}\text{ and }\theta=\arctan\left(\frac{x}{y}\right).$$

We thus conclude that

$$\forall(x,y)\in\mathbb{R}\times\mathbb{R}_+^*,\quad f(x,y)=h(x^2+y^2)\exp\left(\arctan\left(\frac{x}{y}\right)\right).$$

- Conversely, if f is of the form

$$\forall(x,y)\in\mathbb{R}\times\mathbb{R}_+^*,\quad f(x,y)=h(x^2+y^2)\exp\left(\arctan\left(\frac{x}{y}\right)\right),$$

for some function $h:\mathbb{R}_+^*\longrightarrow\mathbb{R}$ of class C^1 then we know, from Question 1 that f is a solution of Equation (E).

Conclusion: the set of solutions of class C^1 of Equation (E) is the set of functions f of the form

$$f(x,y)=h(x^2+y^2)\exp\left(\arctan\left(\frac{x}{y}\right)\right),$$

where $h:\mathbb{R}_+^*\longrightarrow\mathbb{R}$ is of class C^1.

Exercise 2. For $(x,y)\in\mathbb{R}^2$ we have

$$\begin{aligned}d_{(x,y)}\varphi_1&=\cos\theta(x,y)\,dx-\sin\theta(x,y)\,dy\\d_{(x,y)}\varphi_2&=\sin\theta(x,y)\,dx+\cos\theta(x,y)\,dy.\end{aligned}$$

Since $d\varphi_1$ and $d\varphi_2$ are exact differential forms, they are also closed, i.e.,

(1) $\partial_2(\cos\circ\theta)=-\partial_1(\sin\circ\theta)$

(2) $\partial_2(\sin\circ\theta)=\partial_1(\cos\circ\theta)$

Now, from the chain rule, these relations read as

(1) $-\partial_2\theta\,(\sin\circ\theta)=-\partial_1\theta\,(\cos\circ\theta)$

(2) $\partial_2\theta\,(\cos\circ\theta)=-\partial_1\theta\,(\sin\circ\theta)$,

or, in matrix form, for all $(x,y)\in\mathbb{R}^2$,

$$\begin{pmatrix}\cos(\theta(x,y))&-\sin(\theta(x,y))\\\sin(\theta(x,y))&\cos(\theta(x,y))\end{pmatrix}\begin{pmatrix}\partial_1\theta(x,y)\\\partial_2\theta(x,y)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

Hence (since the matrix that appears has a non-nil determinant) we must have, throughout $\mathbb{R}^2$:

$$\partial_1\theta=\partial_2\theta=0.$$

This shows that the differential of θ is nil on $\mathbb{R}^2$, hence (since $\mathbb{R}^2$ is connected), the function θ is constant on $\mathbb{R}^2$. We denote by θ_0 the value of this constant. We hence have, for all $(x,y)\in\mathbb{R}^2$:

$$\begin{aligned}d_{(x,y)}\varphi_1&=\cos\theta_0\,dx-\sin\theta_0\,dy\\d_{(x,y)}\varphi_2&=\sin\theta_0\,dx+\cos\theta_0\,dy.\end{aligned}$$

We then integrate these differential forms and obtain that there exists $(x_0,y_0)\in\mathbb{R}^2$ such that

$$\forall(x,y)\in\mathbb{R}^2,\quad\varphi(x,y)=(x\cos\theta_0-y\sin\theta_0+x_0,x\sin\theta_0+y\cos\theta_0+y_0).$$

Exercise 3.

1. Let $x\in\mathbb{R}$. Then:

$$\begin{aligned}g'(x)=&\Big(\partial_1 f(x,\cos x)-\sin x\,\partial_2 f(x,\cos x)\Big)\partial_1 f\big(f(x,\cos x),f(\cos x,x)\big)\\&+\Big(-\sin x\,\partial_1 f(\cos x,x)+\partial_2 f(\cos x,x)\Big)\partial_2 f\big(f(x,\cos x),f(\cos x,x)\big).\end{aligned}$$

2. For $(x,y)\in\mathbb{R}^*\times\mathbb{R}$,

$$\partial_2 g(x,y)=\frac{1}{x}f'\left(\frac{y}{x}\right)$$

hence

$$\partial^2_{1,2}f(x,y)=-\frac{1}{x^2}f'\left(\frac{y}{x}\right)-\frac{y}{x^3}f''\left(\frac{y}{x}\right).$$

Exercise 4.

1. a) From the Jacobian of φ at $(0,0)$, and since $\varphi(0,0)$ we conclude that

$$\begin{aligned}\partial_1 g(0,0)&=2\partial_1 f(0,0)-\partial_2 f(0,0)\\\partial_2 g(0,0)&=-\partial_1 f(0,0)+2\partial_2 f(0,0).\end{aligned}$$

Hence

$$\begin{aligned}\nabla_{(1,1)}g(0,0)&=\partial_1 g(0,0)+\partial_2 g(0,0)=\partial_1 f(0,0)+\partial_2 f(0,0)=\nabla_{(1,1)}f(0,0)\\\nabla_{(1,-1)}g(0,0)&=\partial_1 g(0,0)-\partial_2 g(0,0)=3\partial_1 f(0,0)-3\partial_2 f(0,0)=3\nabla_{(1,-1)}f(0,0).\end{aligned}$$

b) Since $J_{(0,0)}\varphi$ is the matrix of $D_{(0,0)}\varphi$ is the standard basis of $\mathbb{R}^2$ and since

$$J_{(0,0)}\varphi\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2 & -1\\-1 & 2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$
$$J_{(0,0)}\varphi\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}2 & -1\\-1 & 2\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}3\\-3\end{pmatrix} = 3\begin{pmatrix}1\\-1\end{pmatrix}$$

we conclude that $(1,1)$ is an eigenvector of $D_{(0,0)}\varphi$ associated with the eigenvalue 1 and that $(1,-1)$ is an eigenvector of $D_{(0,0)}\varphi$ associated with the eigenvalue 3.

2. From the chain rule:

$$\mathrm{d}_{x_0}g = \mathrm{d}_{x_0}f \circ D_{x_0}\varphi.$$

Now if $\vec{u}$ is an eigenvector of $D_{x_0}\varphi$ associated with the eigenvalue λ then

$$\nabla_{\vec{u}}g(x_0) = \mathrm{d}_{x_0}g(\vec{u}) = \mathrm{d}_{x_0}f\left(D_{x_0}\varphi(\vec{u})\right) = \mathrm{d}_{x_0}f(\lambda\vec{u}) = \lambda\mathrm{d}_{x_0}f(\vec{u}) = \lambda\nabla_{\vec{u}}f(x_0).$$