INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Math Test #3 — 1h30

January 9, 2015

No documents, no calculators, no cell phones, no axolotl, no naked mole rat allowed but you may keep your pet mantis shrimp for moral support.

All your answers must be fully justified, unless noted otherwise.

Exercise 1. Let $f : \mathbb{R}_+^* \longrightarrow \mathbb{R}$ be a function such that

$$\forall x \in \mathbb{R}_+^*,\ \frac{2x}{1+x} \leq f(x) \leq \sqrt{x}.$$

1. Deduce the values of the limit

$$\lim_{x \to 0^+} f(x) \qquad \text{and} \qquad \lim_{x \to +\infty} \frac{f(x)}{x}.$$

2. Is it possible that (no justifications required):

   a) $\lim\limits_{x \to +\infty} f(x) = +\infty$?

   b) $\lim\limits_{x \to +\infty} f(x) = 0$?

   c) $\lim\limits_{x \to +\infty} f(x)$ DNE?

   d) $\lim\limits_{x \to +\infty} f(x) = 42$?

3. Show that f is continuous at 1.

4. Show that f is differentiable at 1 and determine $f'(1)$.

$$\begin{array}{l|l} 4\ell^3 - a\ell^2 - 2a^2\ell - a^3 & \ell - a \\ -4\ell^3 + 4a\ell^2 & 4\ell^2 + 3a\ell + a^2 \\ \hline \quad 3a\ell^2 - 2a^2\ell & \\ \quad -3a\ell^2 + 3a^2\ell & \\ \hline \qquad a^2\ell - a^3 & \end{array}$$

$$\frac{\ell'}{\ell} = \frac{2\sqrt{\ell\ell'}}{\ell + \ell'}$$

$$\ell'(\ell + \ell') = 2\ell\sqrt{\ell\ell'}$$

$$\ell'^2(\ell + \ell')^2 = 4\ell^3\ell'$$

$$\ell'(\ell^2 + 2\ell\ell' + \ell'^2) = 4\ell^3$$

Exercise 2 (*Geometrico-Harmonic Mean*). Let $a, b \in \mathbb{R}_+^*$ such that $a < b$. We define the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ as follows:

$$x_0 = a, \quad y_0 = b, \quad \forall n \in \mathbb{N},\ x_{n+1} = \frac{2x_n y_n}{x_n + y_n},\ y_{n+1} = \sqrt{x_n y_n}.$$

Your are given:

- the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are well-defined, and for all $n \in \mathbb{N}$, $x_n > 0$ and $y_n > 0$.
- for all $\alpha, \beta \in \mathbb{R}_+^*$ such that $\alpha \neq \beta$, one has:

$$\frac{2\sqrt{\alpha\beta}}{\alpha + \beta} < 1.$$

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1. Show by induction that for all $n \in \mathbb{N}$, $x_n < y_n$.
2. Show that the sequence $(x_n)_{n\in\mathbb{N}}$ is increasing and that the sequence $(y_n)_{n\in\mathbb{N}}$ is decreasing.
3. Deduce that the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ converge, and then prove that they have the same limit. Hint: do not try to show that they are adjacent sequences.

Exercise 3.

1. Show that there exists a unique element $x_0 \in [0,1]$ such that $x_0 = \cos(x_0)$.
2. We define the sequence $(u_n)_{n\in\mathbb{N}}$ as

$$u_0 = 1, \quad \forall n \in \mathbb{N},\ u_{n+1} = \cos u_n.$$

   You are given that for all $n \in \mathbb{N}$, $u_n \in [0,1]$.

   The goal of this question is to show that the sequence $(u_n)_{n\in\mathbb{N}}$ converges to x_0. We define the sequence $(b_n)_{n\in\mathbb{N}}$ as

$$\forall n \in \mathbb{N},\ b_n = u_n - x_0.$$

   We recall that x_0 is the unique element in $[0,1]$ such that $x_0 = \cos x_0$.

   a) Show that for all $n \in \mathbb{N}$,

$$b_{n+1} = -2\sin\left(x_0 + \frac{b_n}{2}\right)\sin\left(\frac{b_n}{2}\right).$$

   b) Explain why

$$\forall n \in \mathbb{N},\ x_0 + \frac{b_n}{2} \in [0,1]$$

   and deduce that

$$0 \leq \sin\left(x_0 + \frac{b_n}{2}\right) \leq \sin 1.$$

   c) Use fact that $\forall x \in \mathbb{R},\ |\sin x| \leq |x|$ to show that

$$\forall n \in \mathbb{N},\ |b_{n+1}| \leq |b_n| \sin 1.$$

   d) Deduce that the sequence $(b_n)_{n\in\mathbb{N}}$ converges to 0, and conclude.

   e) Explain why

$$\forall n \in \mathbb{N},\ |u_n - x_0| \leq \sin^n(1).$$

Exercise 4. Recall the Extreme Value Theorem. Give an example of a function that is defined on $[0,1]$, that is bounded, but that doesn't attain its lower bound and its upper bound.