INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Solution of Math Test #3

January 9, 2015

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Exercise 1.

1. Since $\lim_{x\to 0^+} 2x1+x=0$ and $\lim_{x\to0^+}\sqrt{x}=0$ we conclude, by the Squeeze Theorem, that $\lim_{x\to0^+} f(x)=0$.
   For all $x\in\mathbb{R}_+^*$,
   $$\frac{2}{1+x}\le\frac{f(x)}{x}\le\frac{1}{\sqrt{x}}.$$
   Since $\lim_{x\to+\infty}\frac{2}{1+x}=0$ and $\lim_{x\to+\infty}\frac{1}{\sqrt{x}}=0$ we conclude, by the Squeeze Theorem, that $\lim_{x\to+\infty}\frac{f(x)}{x}=0$.
2. a) Yes.
   b) No.
   c) Yes.
   d) Yes.
3. From the given inequality,
   $$1\le f(1)\le 1,$$
   hence $f(1)=1$. Now, since $\lim_{x\to1}\frac{2x}{1+x}=1$ and $\lim_{x\to1}\sqrt{x}=1$ we conclude, by the Squeeze Theorem, that $\lim_{x\to1} f(x)=1=f(1)$, hence f is continuous at 1.
4. From the given inequality, and since $f(1)=1$, we have:
   $$\forall x\in\mathbb{R}_+^*,\ \frac{2x}{1+x}-1=\frac{x-1}{x+1}\le f(x)-f(1)\le\sqrt{x}-1.$$
   Hence,
   $$\forall x\in(1,+\infty),\ \frac{1}{x+1}\le\frac{f(x)-f(1)}{x-1}\le\frac{\sqrt{x}-1}{x-1}=\frac{1}{\sqrt{x}+1},$$
   hence, since $\lim_{x\to1}\frac{1}{x+1}=\frac12$ and $\lim_{x\to1}\frac{1}{\sqrt{x}+1}=\frac12$ we conclude, by the Squeeze Theorem, that
   $$\lim_{x\to1^+}\frac{f(x)-f(1)}{x-1}=\frac12.$$
   Similarly,
   $$\forall x\in(0,1),\ \frac{1}{\sqrt{x}+1}\le\frac{f(x)-f(1)}{x-1}\le\frac{1}{x+1},$$
   and from the Squeeze Theorem we conclude that
   $$\lim_{x\to1^-}\frac{f(x)-f(1)}{x-1}=\frac12.$$
   Since the left-sided and right-sided limits of $\frac{f(x)-f(1)}{x-1}$ as x approaches 1 are equal to $\frac12$, we conclude that
   $$\lim_{x\to1}\frac{f(x)-f(1)}{x-1}=\frac12,$$
   hence f is differentiable at 1 and $f'(1)=1/2$.

Exercise 2.

1. We have $x_0=a<b=y_0$. We now assume that, for some $n\in\mathbb{N}$, $x_n<y_n$. Then, since $y_{n+1}>0$:
   $$\frac{x_{n+1}}{y_{n+1}}=\frac{2\sqrt{x_ny_n}}{x_n+y_n}<1,$$
   hence, $x_{n+1}<y_{n+1}$. By mathematical induction, we conclude that $\forall n\in\mathbb{N}$, $x_n<y_n$.
2. Let $n\in\mathbb{N}$. Then
   $$\frac{x_{n+1}}{x_n}=\frac{2x_n}{x_n+y_n}>\frac{x_n+y_n}{x_n+y_n}=1,$$
   hence, since the sequence $(x_n)_{n\in\mathbb{N}}$ takes positive values, we deduce that the sequence $(x_n)_{n\in\mathbb{N}}$ is increasing. Also,
   $$\frac{y_{n+1}}{y_n}=\sqrt{\frac{x_n}{y_n}}<1$$
   by Question 1. Hence the sequence $(y_n)_{n\in\mathbb{N}}$ is increasing.
3. By Question 1, and since the sequence $(y_n)_{n\in\mathbb{N}}$ is decreasing, we conclude that for all $n\in\mathbb{N}$, $x_n<y_n\le b$. This shows that the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded from above. We also know that the sequence $(x_n)_{n\in\mathbb{N}}$ is increasing hence, by the Monotone Limit Theorem, we conclude that the sequence $(x_n)_{n\in\mathbb{N}}$ converges. We denote by ℓ its limit. Since the sequence $(x_n)_{n\in\mathbb{N}}$ is increasing, we know that $\ell\ge x_0=a>0$.
   We also know that the sequence $(y_n)_{n\in\mathbb{N}}$ is decreasing and bounded from below (since 0 is a lower bound of $(y_n)_{n\in\mathbb{N}}$), hence the sequence $(y_n)_{n\in\mathbb{N}}$ converges. We denote by ℓ' its limit. Of course, we know that $\ell'\ge0$.
   We now show that $\ell=\ell'$: by the elementary operations on limits (and since we know that $\ell+\ell'\neq0$), we must have:
   $$\ell=\frac{2\ell\ell'}{\ell+\ell'}.$$
   Hence, since $\ell\neq0$,
   $$1=\frac{2\ell'}{\ell+\ell'},$$
   hence $\ell+\ell'=2\ell'$ hence $\ell=\ell'$.

Exercise 3.

1. Define the function g as
   $$g:\ \begin{array}{ccc}[0,1] & \longrightarrow & \mathbb{R}\\ x & \longmapsto & x-\cos(x).\end{array}$$
   Clearly, the function g is continuous on $[0,1]$, $g(0)=-1<0$ and $g(1)=1-\cos(1)>0$ hence, by the Intermediate Value Theorem, there exists an element $x_0\in[0,1]$ such that $g(x_0)=0$, i.e., $x_0=\cos(x_0)$. Since g is increasing (as the sum of two increasing functions), we know that g is $1-1$; hence such an x_0 is unique.
2. a) Let $n\in\mathbb{N}$:
   $$\begin{aligned}b_{n+1}&=u_{n+1}-x_0\\&=\cos(u_n)-\cos(x_0)\qquad\text{since } x_0=\cos(x_0)\\&=-2\sin\left(\frac{u_n+x_0}{2}\right)\sin\left(\frac{u_n-x_0}{2}\right)\\&=-2\sin\left(\frac{b_n+2x_0}{2}\right)\sin\left(\frac{b_n}{2}\right).\end{aligned}$$
   b) Let $n\in\mathbb{N}$. On the one hand,
   $$x_0+\frac{b_n}{2}=x_0+\frac{u_n-x_0}{2}=\frac{u_n+x_0}{2},$$
   and on the other hand, since $0\le x_0\le1$ and $0\le u_n\le1$, one has
   $$0\le\frac{u_n+x_0}{2}\le1,$$
   hence the result.
   Since $1<\pi/2$ and since sin is increasing on $[0,\pi/2]$, we conclude that
   $$0\le\sin\left(x_0+\frac{b_n}{2}\right)\le\sin1.$$
   c) Let $n\in\mathbb{N}$. From the equality of Question b and the given inequality, we have:
   $$|b_n|=2\sin\left(x_0+\frac{b_n}{2}\right)\left|\sin\left(\frac{b_n}{2}\right)\right|\le2\frac{b_n}{2}\sin1=b_n\sin1.$$

d) By mathematical induction, it is straightforward to obtain the following result:

$$\forall n \in \mathbb{N},\ |b_n| \leq b_0 \sin^n(1).$$

Since $0 < \sin(1) < 1$, we have: $\lim\limits_{n\to+\infty} \sin^n(1) = 0$ hence we conclude, by the Squeeze Theorem, that $\lim\limits_{n\to+\infty} |b_n| = 0$, hence $\lim\limits_{n\to+\infty} b_n = 0$.
Since for all $n \in \mathbb{N}$, $u_n = x_0 + b_n$, we conclude that $\lim\limits_{n\to+\infty} u_n = x_0$.

e) The inequality shown in Question d also reads as

$$\forall n \in \mathbb{N},\ |u_n - x_0| \leq |1 - x_0| \sin^n(1).$$

Since $0 \leq x_0 \leq 1$, $0 \leq 1 - x_0 \leq 1$, hence $|1 - x_0| \leq 1$ hence

$$\forall n \in \mathbb{N},\ |u_n - x_0| \leq \sin^n(1).$$

Exercise 4. Extreme Value Theorem: every continuous function defined on a closed and bounded set is bounded and attains its bounds.
Let f be the function defined by

$$f : [0,1] \longrightarrow \mathbb{R}$$
$$x \longmapsto \begin{cases} x & \text{if } x \in (0,1) \\ \frac{1}{2} & \text{if } x \in \{0,1\}. \end{cases}$$

The function f is clearly bounded:

$$\inf_{[0,1]} f = 0 \quad \text{and} \sup_{[0,1]} f = 1,$$

yet f never attains its lower bound and its upper bound.