INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Math Test #2 — 1h30

November 28, 2014

No documents, no calculators, no cell phones, no axolotl allowed but you may keep your pet naked mole rat for moral support.

All your answers must be fully justified, unless noted otherwise.

Exercise 1. The goal of this exercise is to prove, using the ε–δ definition of a limit that

$$\lim_{x\to 2} x^2 = 4.$$

1. Recall the ε–δ definition of $\lim_{x\to 2} x^2 = 4$.
2. Let $\delta > 0$ and $x \in \mathbb{R}$. Prove that

$$|x-2| < \delta \implies |x+2| < \delta + 4.$$

   and deduce that

$$|x-2| < \delta \implies \left|x^2 - 4\right| < \delta(\delta+4).$$

3. Let $\varepsilon > 0$ and set $\delta = \sqrt{\varepsilon + 4} - 2$. Briefly check that δ is well-defined and that $\delta > 0$. Deduce from the previous questions that

$$|x-2| < \delta \implies \left|x^2 - 4\right| < \varepsilon$$

   and conclude.

Exercise 2 (*Chebyshev Polynomials*). The Chebyshev polynomial functions are defined inductively by:

$$T_0 : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & 1, \end{array} \qquad T_1 : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & x, \end{array} \qquad \forall n \in \mathbb{N},\quad T_n : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & 2xT_{n-1}(x) - T_{n-2}(x). \end{array}$$

You are given that for all $n \in \mathbb{N}$, the polynomial function T_n is of degree n; you may use this fact without any justifications.

$2\cos((n+1)x)\cos x + \sin((n+1)x)\sin(x) -$

1. For $x \in \mathbb{R}$, explicit $T_2(x)$ and $T_3(x)$.
2. Show that for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$,

$$2\cos(x)\cos\big((n+1)x\big) - \cos(nx) = \cos\big((n+2)x\big).$$

3. Show that for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$,

$$T_n(\cos x) = \cos(nx).$$

   This question should be solved by mathematical induction. You will clearly show where you're using the inductive hypothesis.

4. Deduce that, for all $n \in \mathbb{N}^*$, the roots of the polynomial T_n are the real numbers

$$\cos\left(\frac{(2k+1)\pi}{2n}\right), \quad \text{with } k \in \{0, \dots, n-1\}$$

   and that these roots are simple roots.

5. You're given that

$$\forall n \in \mathbb{N},\ \forall x \in \mathbb{R}, \quad T_n(\cosh x) = \cosh(nx),$$

   and you may use this fact without any justifications. Deduce that for all $m, n \in \mathbb{N}^*$,

$$T_m \circ T_n = T_{mn}.$$

   Hint: first show that for all $t \in [1, +\infty)$, $T_m\big(T_n(t)\big) = T_{mn}(t)$. Hint for the hint: $[1, +\infty)$ is the range of a well-known hyperbolic function.

$|t| \in [1, +\infty$

Exercise 3. Define the function f by

$$f : \mathbb{R}_+ \longrightarrow (0,1]$$
$$x \longmapsto \frac{1}{\cosh x}.$$

1. Show that the function f is well-defined.
2. Determine the variations of f.
3. Show that f is a bijection and determine f^{-1} explicitly.
4. Determine (without any justifications), if they exist, the value of

$$\sup_{\mathbb{R}_+} f, \qquad \max_{\mathbb{R}_+} f, \qquad \inf_{\mathbb{R}_+} f, \qquad \min_{\mathbb{R}_+} f.$$

In case of non-existence, write DNE.

$$\frac{1}{\cosh(x')} - \frac{1}{\cosh(x)}$$
$$= \frac{\cosh(x) - \cosh(x')}{\cosh(x) \cdot \cosh(x')}$$
$$= \frac{\cosh\left(\frac{x+x'}{2} + \frac{x-x'}{2}\right) - \cosh\left(\frac{x+x'}{2} - \frac{x-x'}{2}\right)}{\cosh(x) \cdot \cosh(x')}$$
$$= \cosh\left(\frac{x+x'}{2}\right.$$

Exercise 4. Show that for all $n \in \mathbb{N}^*$,

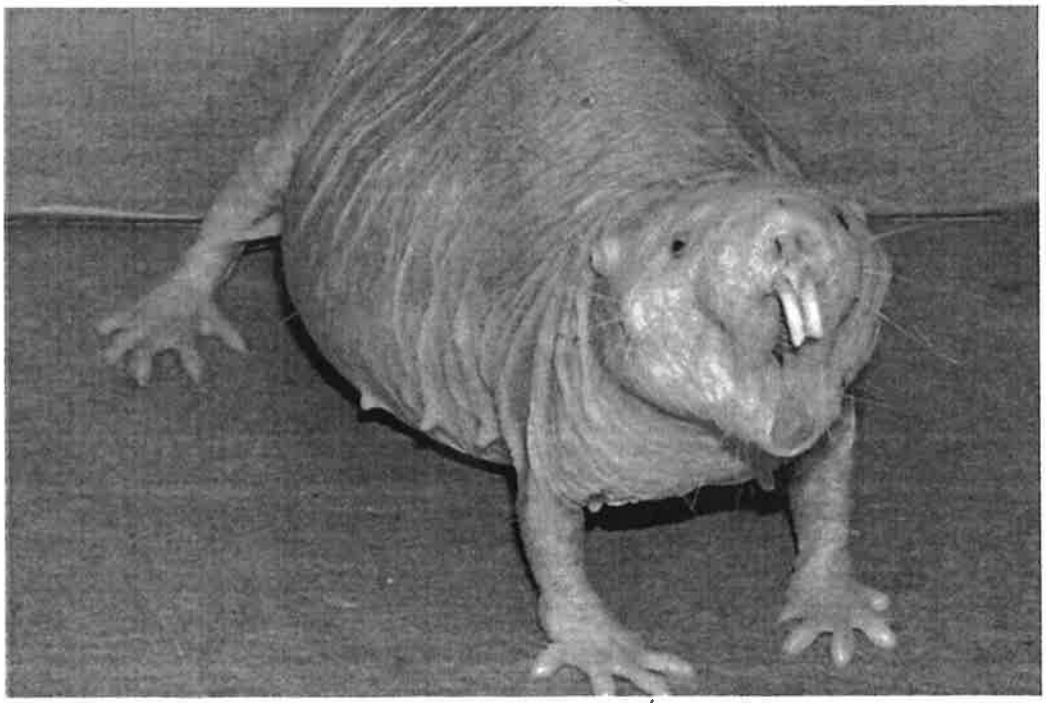
$$\sum_{k=1}^{n} \frac{1}{k^2} < 2.$$

You may use without any justifications the following fact:

$$\forall k \geq 2, \quad \frac{1}{k^2} \leq \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}.$$

Be extremely careful: this fact is only valid for $k \geq 2$.

$$\sum_{k=1}^{n} \frac{1}{k^2} = 1 + \sum_{k=2}^{n} \frac{1}{k^2}$$
$$\leq 1 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right)$$
$$\leq 1 + \left(\sum_{k=1}^{n-1} \frac{1}{k}\right) - \sum_{k=2}^{n} \frac{1}{k}$$



$$\leq 1 + 1 - \frac{1}{n}$$
$$\leq$$

$$\frac{1-\sqrt{1-y^2}}{y} < 1 \qquad \frac{1}{y} - \frac{\sqrt{1-y^2}}{y}$$

$$\cancel{1 = 1} \cdot \frac{y^2}{y(1+\sqrt{1-y^2})} = \frac{y}{(}$$