INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Solution of Math Test #2

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Romaric Pujol, romaric.pujol@insa-lyon.fr

Exercise 1.

1. $\lim_{x\to 2} x^2 = 4$ means:
$$\forall \varepsilon>0,\ \exists\delta>0,\ \forall x\in\mathbb{R},\ \Big(0<|x-2|<\delta \implies \left|x^2-4\right|<\varepsilon\Big).$$

2. Let $\delta>0$ and let $x\in\mathbb{R}$ such that $|x-2|<\delta$. Then, by the triangle inequality:
$$|x+2| = |x-2+4| \le |x-2|+4 < \delta+4.$$
Hence,
$$\left|x^2-4\right| = |x-2||x+2| < \delta(\delta+4).$$

3. Let $\varepsilon>0$ and $\delta=\sqrt{\varepsilon+4}-2$. Since $\varepsilon>0$, $\varepsilon+4>4>0$, hence δ is well-defined (we're taking the square root of a positive number) and $\sqrt{\varepsilon+4}>\sqrt{4}=2$, hence $\delta>0$.
Now, let $x\in\mathbb{R}$ such that $|x-2|<\delta=\sqrt{\varepsilon+4}-2$. Then:
$$\left|x^2-4\right| < \delta^2+4\delta = \left(\sqrt{\varepsilon+4}-2\right)^2 + 4\left(\sqrt{\varepsilon+4}-2\right) = \left(\varepsilon+4-4\sqrt{\varepsilon+4}+4\right)+4\left(\sqrt{\varepsilon+4}-2\right)=\varepsilon.$$
We have thus shown that for all $\varepsilon>0$, there exists $\delta>0$, namely $\delta=\sqrt{\varepsilon+4}-2$, such that for all $x\in\mathbb{R}$ such that $0<|x-2|<\delta$ one has $\left|x^2-4\right|<\varepsilon$, i.e., $\lim_{x\to 2}x^2=4$.

Exercise 2.

1. Let $x\in\mathbb{R}$:
$$T_2(x) = 2xT_1(x)-T_0(x) = 2x^2-1,$$
$$T_3(x) = 2xT_2(x)-T_1(x) = 2x(2x^2-1)-x = 4x^3-3x.$$

2. Let $x\in\mathbb{R}$ and $n\in\mathbb{N}$: by the transformation of sum into product:
$$\begin{aligned}2\cos(x)\cos\big((n+1)x\big)-\cos(nx) &= \cos\big(x+(n+1)x\big)+\cos\big(x-(n+1)x\big)-\cos(nx)\\ &= \cos\big((n+2)x\big)+\cos(-nx)-\cos(nx)\\ &= \cos\big((n+2)x\big) \qquad \textit{since cos is even.}\end{aligned}$$

3. Let $x\in\mathbb{R}$. For $n\in\mathbb{N}$ we define the proposition (P_n) by
$$(P_n)\qquad \begin{cases} T_n(\cos x)=\cos(nx)\\ T_{n+1}(\cos x)=\cos\big((n+1)x\big).\end{cases}$$
   - The proposition (P_0) is true, since
   $$T_0(\cos x)=1=\cos(0x) \quad\text{and}\quad T_1(\cos x)=\cos x.$$
   - We assume that, for some $n\in\mathbb{N}$, the proposition (P_n) is true. Then, obviously, $T_{n+1}(\cos x)=\cos\big((n+1)x\big)$ (this is the second relation in (P_n)), i.e., the first relation of (P_{n+1}) is true. Now:
   $$\begin{aligned}T_{n+2}(\cos x) &= 2\cos(x)T_{n+1}(\cos x)-T_n(\cos x)\\ &= 2\cos(x)\cos\big((n+1)x\big)-\cos(nx) && \textit{we used both relations of } (P_n)\\ &= \cos\big((n+2)x\big) && \textit{by Question 2.}\end{aligned}$$
   Hence the second relation of (P_{n+1}) is true; hence (P_{n+1}) is true.

Hence, for all $n\in\mathbb{N}$, and for all $x\in\mathbb{R}$,
$$T_n(\cos x)=\cos(nx).$$

4. Let $n\in\mathbb{N}^*$ and let $k\in\{0,\dots,n-1\}$. Then, by the result of the previous question:
$$\begin{aligned}T_n\left(\cos\left(\frac{(2k+1)\pi}{2n}\right)\right) &= \cos\left(\frac{(2k+1)\pi}{2}\right)\\ &= \cos\left(\frac{\pi}{2}+k\pi\right)\\ &= 0.\end{aligned}$$
This shows that $\cos\left(\frac{(2k+1)\pi}{2n}\right)$ is a root of T_n.

Now, since cos is decreasing (hence 1 − 1) on $[0,\pi]$ and since $\frac{(2k+1)\pi}{2n}\in[0,\pi]$, we conclude that there are exactly n elements (all distinct) of the form
$$(*)\qquad \cos\left(\frac{(2k+1)\pi}{2n}\right),\ k\in\{0,\dots,n-1\}.$$
Since $\deg T_n=n$, T_n can't have more than n roots (counted with their multiplicities); hence the roots of T_n are exactly the numbers described in $(*)$, and these roots are of multiplicity 1.

5. Let $m,n\in\mathbb{N}^*$ and let $t\in[1,+\infty)$. We'll use the fact that $t=\cosh(\operatorname{arccosh} t)$, which is valid since $\operatorname{arccosh} t$ is well-defined (since t is in the domain of arccosh). Then:
$$\begin{aligned}T_m\big(T_n(t)\big) &= T_m\Big(T_n\big(\cosh(\operatorname{arccosh} t)\big)\Big)\\ &= T_m\big(\cosh(n\operatorname{arccosh} t)\big)\\ &= \cosh(mn\operatorname{arccosh} t)\\ &= T_{mn}\big(\cosh(\operatorname{arccosh} t)\big)\\ &= T_{mn}(t).\end{aligned}$$
Now, the function $T_m\circ T_n - T_{mn}$ is a polynomial function that has an infinite number of roots (as we showed that all the elements in $[1,+\infty)$ are roots), hence this polynomial function is the nil function:
$$T_m\circ T_n - T_{mn} = 0_{\mathbb{R}},$$
i.e.,
$$T_m\circ T_n = T_{mn}.$$

Exercise 3.

1. Let $x\in\mathbb{R}_+$. We know that $\cosh x\ge 1$, hence $0<\frac{1}{\cosh x}\le 1$, hence f is well-defined.

2. We know that the function cosh takes values in $\mathbb{R}_+^*$ and is increasing on $\mathbb{R}_+$. We also know that the function $x\mapsto \frac{1}{x}$ is decreasing on $\mathbb{R}_+^*$. Hence, by composition, f is decreasing.

3. Let $x\in\mathbb{R}_+$ and $y\in(0,1]$. Then, observing that $\frac{1}{y}\ge 1$ (hence $\frac{1}{y}$ is in the domain of arccosh), we obtain:
$$y=f(x) \iff y=\frac{1}{\cosh(x)} \iff \cosh x=\frac{1}{y} \iff x=\operatorname{arccosh}\left(\frac{1}{y}\right).$$
This equivalence shows that f is a bijection and that
$$f^{-1}:\ \begin{array}{rcl}(0,1] & \longrightarrow & \mathbb{R}_+\\ y & \longmapsto & \operatorname{arccosh}\left(\frac{1}{y}\right).\end{array}$$

4.

$$\sup_{\mathbb{R}_+} f = 1, \qquad \max_{\mathbb{R}_+} f = 1, \qquad \inf_{\mathbb{R}_+} f = 0, \qquad \min_{\mathbb{R}_+} f \text{ DNE.}$$

Exercise 4. We clearly have

$$\sum_{k=1}^{1} \frac{1}{k^2} = 1 \leq 2.$$

Now, if $n \in \mathbb{N}^*$ with $n \geq 2$,

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k^2} &= 1 + \sum_{k=2}^{n} \frac{1}{k^2} \\
&\leq 1 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k} \right) \\
&= 1 + \sum_{k=2}^{n} \frac{1}{k-1} - \sum_{k=2}^{n} \frac{1}{k} \\
&= 1 + \sum_{\ell=1}^{n-1} \frac{1}{\ell} - \sum_{k=2}^{n} \frac{1}{k} \\
&= 1 + 1 - \frac{1}{n} \\
&= 2 - \frac{1}{n} \\
&< 2.
\end{aligned}$$