INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Math Test #1 — 1h30

October 24, 2014

No documents, no calculators, no cell phones allowed but you may keep your pet axolotl for moral support. All your answers must be fully justified, unless noted otherwise.

Exercise 1. Find all the solutions in $[-2\pi, 2\pi]$ of the following equation in x:

$$1 - \sqrt{2}\cos(x) + \cos(2x) = 0.$$

Exercise 2. Let f be the function defined[1] by

$$f : [0,2] \longrightarrow [0,2]$$
$$x \longmapsto \begin{cases} x^2 & \text{if } x \in [0,1) \\ 0 & \text{if } x = 1 \\ 4-2x & \text{if } x \in (1,2]. \end{cases}$$

1. Sketch the graph of f.
2. Determine (no justifications required):

$$f([0,2]), \quad f((1,2]), \quad f((0,1]), \quad f([1/2,3/2]),$$
$$f^{[-1]}([0,2]), \quad f^{[-1]}([0,1]), \quad f^{[-1]}(\{1\}), \quad f^{[-1]}(\{2\}).$$

3. Show that the function f is decreasing on $(1,2]$.
4. Is the function f injective? surjective? bijective? (Justify).

Exercise 3. Let $q \in \mathbb{C} \setminus \{1\}$ and $a \in \mathbb{C}$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers such that:

$$\forall n \in \mathbb{N}, \quad u_{n+1} = qu_n + a.$$

We set

$$r = \frac{a}{1-q},$$

and we define the sequence $(v_n)_{n\in\mathbb{N}}$ by

$$\forall n \in \mathbb{N}, \quad v_n = u_n - r.$$

1. Show that the sequence $(v_n)_{n\in\mathbb{N}}$ is a geometric sequence of ratio q.
2. Deduce that

$$\forall n \in \mathbb{N}, \quad u_n = q^n(u_0 - r) + r.$$

3. Let $N \in \mathbb{N}$. Find an expression of the following sum that doesn't involve the $\sum$ symbol (and no ellipses either!).

$$\sum_{n=0}^{N} u_n.$$

[1] You don't need to justify that f is well-defined: we have checked that for you.

Exercise 4. Let $\alpha, \beta \in \mathbb{R}_+^*$. We define the sequence $(v_n)_{n \in \mathbb{N}}$ by:

$$\begin{cases} v_0 = 0 \\ v_1 = 1 \\ \forall n \in \mathbb{N}, \quad v_{n+2} = \alpha v_{n+1} + \beta v_n. \end{cases}$$

You may use, without any justifications, that the sequence $(v_n)_{n \in \mathbb{N}}$ is well-defined.

1. Show that the equation
$$X^2 = \alpha X + \beta$$
possesses two distinct real solutions. We denote these solutions by φ and ψ.

2. Show that:
$$\forall n \in \mathbb{N}, \quad v_n = \frac{\varphi^n - \psi^n}{\varphi - \psi}.$$

3. In the case $\alpha > 1$, show that the sequence $(v_n)_{n \in \mathbb{N}}$ is increasing.

`#mathblowing.  Is there such thing as "the smallest positive real number?"`

By positive, *we mean* > 0. *You must justify your answer.*