INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON

SCAN 1 — Solution of Math Test #1

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Exercise 1. Observe that for $x \in \mathbb{R}$,

$$1 - \sqrt{2}\cos(x) + \cos(2x) = -\sqrt{2}\cos(x) + 2\cos^2(x) = \cos(x)\left(-\sqrt{2} + 2\cos(x)\right).$$

We now solve the equation in $\mathbb{R}$. Let $x \in \mathbb{R}$:

$$\begin{aligned} 1 - \sqrt{2}\cos(x) + \cos(2x) = 0 &\iff \cos(x) = 0 \quad\text{or}\quad \cos(x) = \frac{\sqrt{2}}{2} \\ &\iff \exists k \in \mathbb{Z},\quad \left(x = \frac{\pi}{2} + k\pi \quad\text{or}\quad x = \frac{\pi}{4} + 2k\pi \quad\text{or}\quad x = -\frac{\pi}{4} + 2k\pi\right). \end{aligned}$$

We now select the solutions that lie in $[-2\pi, 2\pi]$:

- for the solutions of the form $\frac{\pi}{2} + k\pi$: let $k \in \mathbb{Z}$:

$$\begin{aligned} \frac{\pi}{2} + k\pi \in [-2\pi, 2\pi] &\iff -2\pi \le \frac{\pi}{2} + k\pi \le 2\pi \\ &\iff -2 \le \frac{1}{2} + k \le 2 \\ &\iff -\frac{5}{2} \le k \le \frac{3}{2} \\ &\iff k \in \{-2, -1, 0, 1\}. \end{aligned}$$

  This corresponds to the solutions

$$-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}.$$

- for the solutions of the form $\frac{\pi}{4} + 2k\pi$: let $k \in \mathbb{Z}$:

$$\begin{aligned} \frac{\pi}{4} + 2k\pi \in [-2\pi, 2\pi] &\iff -2\pi \le \frac{\pi}{4} + 2k\pi \le 2\pi \\ &\iff -2 \le \frac{1}{4} + 2k \le 2 \\ &\iff -\frac{9}{4} \le 2k \le \frac{7}{4} \\ &\iff -\frac{9}{8} \le k \le \frac{7}{8} \\ &\iff k \in \{-1, 0\}. \end{aligned}$$

  This corresponds to the solutions

$$-\frac{7\pi}{4}, \frac{\pi}{4}.$$

- for the solutions of the form $-\frac{\pi}{4} + 2k\pi$: let $k \in \mathbb{Z}$:

$$\begin{aligned} -\frac{\pi}{4} + 2k\pi \in [-2\pi, 2\pi] &\iff -2\pi \le -\frac{\pi}{4} + 2k\pi \le 2\pi \\ &\iff -2 \le -\frac{1}{4} + 2k \le 2 \\ &\iff -\frac{7}{4} \le 2k \le \frac{9}{4} \\ &\iff -\frac{7}{8} \le k \le \frac{9}{8} \\ &\iff k \in \{0, 1\}. \end{aligned}$$

  This corresponds to the solutions

$$-\frac{\pi}{4}, \frac{7\pi}{4}.$$

Finally, the set of solutions is:

$$\left\{-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, -\frac{7\pi}{4}, \frac{\pi}{4}, -\frac{\pi}{4}, \frac{7\pi}{4}\right\}.$$

Exercise 2.

1. See Figure 1.

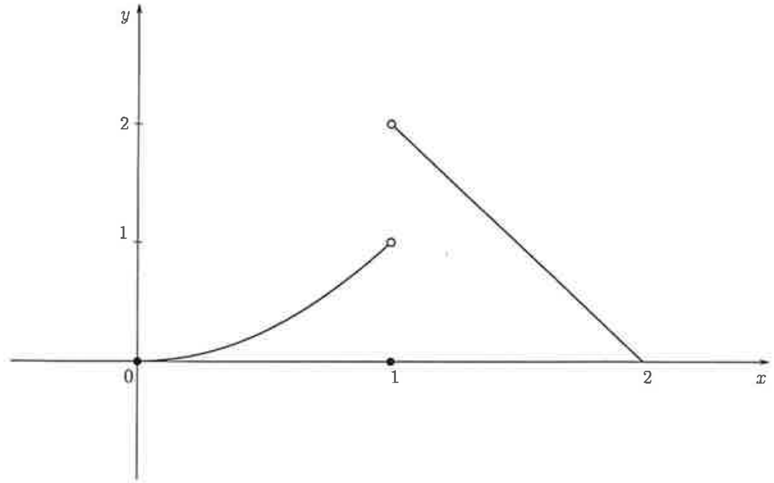

Figure 1: Graph of the function of Exercise 2.

2. From the graph of f:

$$f([0,2]) = [0,2), \quad f((1,2]) = [0,2), \quad f((0,1]] = [0,1), \quad f([1/2,3/2]) = \{0\} \cup [1/4,2),$$
$$f^{[-1]}([0,2]) = [0,2], \quad f^{[-1]}([0,1]) = [0,1] \cup [3/2,2], \quad f^{[-1]}(\{1\}) = \{3/2\}, \quad f^{[-1]}(\{2\}) = \emptyset.$$

3. Let $x, y \in (1,2]$ with $x < y$. Then:

$$\begin{aligned} &-2x > -2y \\ \text{hence}\quad &4 - 2x > 4 - 2y \\ \text{hence}\quad &f(x) > f(y). \end{aligned}$$

   Hence f is decreasing on $(1,2]$.

4. The function f is not injective since $f(0) = f(1) = 0$ and $0 \neq 1$. The function f is not surjective since there are no elements $x \in [0,2]$ such that $f(x) = 2$, as shown by $f^{[-1]}(\{2\}) = \emptyset$. The function f is not bijective.

Exercise 3.

1. Let $n \in \mathbb{N}$:

$$\begin{aligned} v_{n+1} &= u_{n+1} - r \\ &= qu_n + a - \frac{a}{1-q} \\ &= qu_n - \frac{qa}{1-q} \\ &= qu_n - qr \\ &= q(u_n - r) = qv_n. \end{aligned}$$

   Hence the sequence $(v_n)_{n\in\mathbb{N}}$ is a geometric sequence of ratio q.

2. Hence, for $n \in \mathbb{N}$,

$$v_n = q^n v_0 = q^n(u_0 - r),$$

i.e.,

$$u_n = v_n + r = q^n(u_0 - r) + r.$$

3.

$$\begin{aligned}\sum_{n=0}^{N} u_n &= \sum_{n=0}^{N} (q^n(u_0 - r) + r)\\ &= (u_0 - r)\sum_{n=0}^{N} q^n \quad + \quad \sum_{n=0}^{N} r\\ &= (u_0 - r)\frac{1 - q^{N+1}}{1 - q} + (N+1)r.\end{aligned}$$

Exercise 4.

1. You may use the fact that the discriminant of the quadratic

$$X^2 - \alpha X - \beta$$

is $\alpha^2 + 4\beta$ which is positive since $\alpha > 0$ and $\beta > 0$, hence its solutions are real and distinct. Or, you may write this quadratic as:

$$X^2 - \alpha X - \beta = (X - \alpha/2)^2 - \alpha^2/4 - \beta = (X - \alpha/2)^2 - (\alpha^2/4 + \beta)$$

and observe that there are two distinct real solutions since $(\alpha^2/4 + \beta) > 0$ since $\alpha > 0$ and $\beta > 0$.

2. Define the sequence $(u_n)_{n\in\mathbb{N}}$ by

$$\forall n \in \mathbb{N}, \quad u_n = \frac{\varphi^n - \psi^n}{\varphi - \psi}.$$

We'll show by induction that $(u_n)_{n\in\mathbb{N}} = (v_n)_{n\in\mathbb{N}}$. For $n \in \mathbb{N}$ define the proposition (P_n) as:

(P_n) $$v_n = u_n \quad \text{and} \quad v_{n+1} = u_{n+1}.$$

- Initial step: we show that (P_0) is true:

$$u_0 = \frac{\varphi^0 - \psi^0}{\varphi - \psi} = 0 = v_0 \quad \text{and} \quad u_1 = \frac{\varphi^1 - \psi^1}{\varphi - \psi} = 1 = v_1$$

hence (P_0) is true.

- Assume that (P_n) is true for some $n \in \mathbb{N}$. Let's show that (P_{n+1}) is true. Since (P_n) is true, we automatically have $v_{n+1} = u_{n+1}$. Moreover,

$$\begin{aligned}u_{n+2} &= \frac{\varphi^{n+2} - \psi^{n+2}}{\varphi - \psi}\\ &= \frac{\varphi^n\varphi^2 - \psi^n\psi^2}{\varphi - \psi}\\ &= \frac{\varphi^n(\alpha\varphi + \beta) - \psi^n(\alpha\psi + \beta)}{\varphi - \psi}\\ &= \alpha\frac{\varphi^{n+1} - \psi^{n+1}}{\varphi - \psi} + \beta\frac{\varphi^n - \psi^n}{\varphi - \psi}\\ &= \alpha u_{n+1} + \beta u_n\\ &= \alpha v_{n+1} + \beta v_n \qquad \textit{since } (P_n) \textit{ is true}\\ &= v_{n+2}.\end{aligned}$$

In this computation, we have used the facts that $\varphi^2 = \alpha\varphi + \beta$ and $\psi^2 = \alpha\psi + \beta$, since φ and ψ are the solutions of the quadratic

$$X^2 - \alpha X + \beta = 0.$$

Hence (P_{n+1}) is true.

Hence,

$$\forall n \in \mathbb{N}, \quad v_n = u_n = \frac{\varphi^n - \psi^n}{\varphi - \psi}.$$

3. Assume that $\alpha > 1$. For $n \in \mathbb{N}$ define the proposition

(Q_n) $$v_{n+1} > v_n \geq 0.$$

- Initial step: $v_1 = 1 > v_0 = 0 \geq 0$, hence (Q_0) is true.
- Assume that (Q_n) is true for some $n \in \mathbb{N}$. Let's show that $(Q_{n+1}$ is true.

$$\begin{aligned}v_{n+2} &= \alpha v_{n+1} + \beta v_n\\ &\geq \alpha v_{n+1} \qquad \textit{since } v_n \geq 0\\ &> v_{n+1} \qquad \textit{since } v_{n+1} > 0 \textit{ and } \alpha > 1\\ &\geq 0 \qquad \textit{since } (Q_n) \textit{ is true.}\end{aligned}$$

Hence $v_{n+2} > v_{n+1} \geq 0$, hence (Q_{n+1}) is true.

We now conclude that the sequence $(v_n)_{n\in\mathbb{N}}$ is increasing.

`#mathblowing.`
The answer is no, and we can show it by contradiction! Assume that *the smallest positive real number* exists, and denote it by α. Define

$$\beta = \frac{\alpha}{2}.$$

We have:

- $\beta > 0$, since $\alpha > 0$,
- $\beta < \alpha$, again since $\alpha > 0$.

We thus constructed a *positive real number* β *that is smaller than* α. But this is impossible since, supposedly, α is the *smallest positive real number.*