INSTITUT NATIONAL DES SCIENCES APPLIQUÉES DE LYON

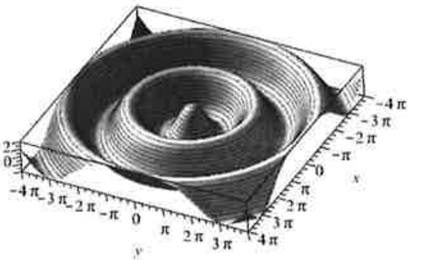
Département du Premier Cycle - SCAN - First

DS 1 MTES - Duration 3 h

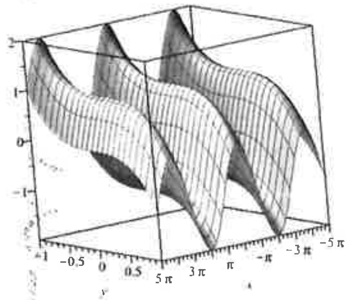
Warnings and advices
– **Documents, dictionaries, phones, and calculators are FORBIDDEN.**
– **The marking scheme is only given as an indication.**

EXERCISE 1 *(2 pts)*

Here are the graph of two functions f and g from $\mathbb{R}^2$ to $\mathbb{R}$.



f(x,y)



g(x, y)

1. Suppose that f is differentiable on its domain, what is the sign of $\frac{\partial f}{\partial y}$ at the points $(4\pi, 4\pi)$ and $(4\pi, -4\pi)$.
2. Suppose that g is differentiable on $\mathbb{R}^2$, find a point $(x_0, y_0) \in \mathbb{R}^2$ such that $\frac{\partial g}{\partial y}(x_0, y_0) = 0$. Is it a minimum or a maximum ?
3. Propose a simple expression for the functions f and g.
4. Draw the graph of the partial map of f with respect to y at the point $(0, 0)$.
5. Draw the graphs of the partial maps of g at the point $(0, 0)$.

EXERCISE 2 *(3 pts)*

We consider the following differential form on $\mathbb{R}^3$:

$$\omega = (y^2(1+x) + \frac{y}{z})dx + \frac{2x\,y\,z + 1}{z}dy - \frac{y}{z^2}dz$$

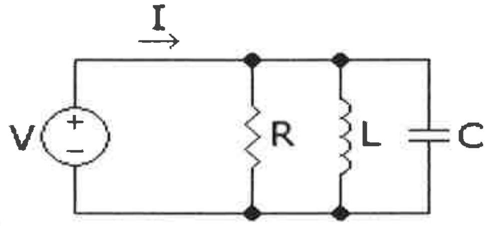
1. Show that ω is not an exact form. Find a function $\phi(x)$ such that :
   - $\phi(0) = 1$
   - $\omega_1 = \phi\omega$ is exact.
2. Find f such that $df = \omega_1$.

Exercise 3 *(3 pts)*

1. (a) Find the complex solutions of the equation : $Z^5 = 1$.
   (b) **Deduce from it** the four solutions of $(z+1)^5 = (z-1)^5$.
   (c) Show that the solutions are real numbers and give an expression without i.
2. (a) Solve the equation : $5z^4 + 10z^2 + 1 = 0$.
   (b) Show that the solutions are exactly the solutions of the equation of question $1)b)$.
3. Deduce from the previous questions the exact value of $tan(\frac{\pi}{5})$.

Exercise 4 *(4.5 pts)*

Consider a parallel RLC electrical circuit with a resistor of resistance R, an inductor of inductance L and a capacitor of capacitance C. The three numbers R, L and C are real positive numbers.



Let $\omega_0 = \sqrt{\frac{1}{LC}}$.

In this case, we consider that the voltage is a complex valued function of the form $U = U_0 e^{i(\omega t + \phi)}$ with $U_0, \omega \in \mathbb{R}^+$ and $\phi \in [0, 2\pi[$.

The current I is also a complex function of the form $I = I_0 e^{i\omega t}$ such that $\frac{dI}{dt} = C\frac{d^2U}{dt^2} + \frac{1}{R}\frac{dU}{dt} + \frac{U}{L}$.

1. The complex impedance of the circuit is the complex number Z such that $U = ZI$. Show that $Z = \frac{1}{\frac{1}{R} + i(C\omega - \frac{1}{L\omega})}$.
2. Compute the module of Z.
3. Compute the argument θ of Z such that $\theta \in]-\pi, \pi]$. Justify rigorously your answer.
4. In the following, we consider that Z is a function of the three variables R, L and C.
   (a) Show that $dZ = \frac{Z^2}{R^2}dR - i\omega Z^2 dC - \frac{iZ^2}{\omega L^2}dL$ and compute the differential $d\overline{Z}$ of $\overline{Z}(R, L, C)$.
   (b) Compute $\overline{Z}dZ + Zd\overline{Z}$.
   (c) Deduce from it the differential of $|Z|^2$.
   (d) During time, the electrical components are slightly modified. Compute the variation $\delta|Z|$ of $|Z|$ at first order with the following numerical values :
   $R = 1\Omega$, $L = 1H$, $C = 1F$, $\omega = \omega_0$ and $\delta R = 1\mu\Omega$, $\delta L = 1\mu H$, $\delta C = -1\mu F$
   (e) The manufacturer indicates that $R = 1\Omega, L = 1H$ and $C = 1F$ are known with the following uncertainties : $\Delta R = 2\%$, $\Delta L = \Delta C = 1\%$. Compute the uncertainty on the argument of Z when $\omega = \omega_0$.

Exercise 5 *(4.5 pts)*

Consider the space $\mathbb{R}^3$ with an orthonormal frame and the points $A(1,1,0), B(-1,0,1), C(1,-1,1)$ and $M(1,2,1)$. Let H be the orthogonal projection of M on the plane (ABC).

1. Compute the area of the triangle ABC.
2. Give the cartesian equation of the plane (ABC).
3. Find the coordinates of a vector $\vec{u}$ such that $\vec{u}$ is a unitary vector with direction the line (HM) and such that $\vec{u}$ goes from H to M.
4. Using a scalar product, compute the distance HM.
5. Compute the coordinates of the point M.
6. Let G be the center of mass of all points A, B, C, M. Compute the coordinates of G.
7. Let G_1 be the center of mass of A, B, C.
   (a) Show that G_1 belongs to the plane (ABC).
   (b) Prove that the points M, G, G_1 are aligned and that G is between M and the plane (ABC).
8. Let G_2 be the orthogonal projection of G on the plane ABC.
   (a) Find two real numbers $\lambda, \mu \in \mathbb{R}$ such that $\overrightarrow{G_2H}+\overrightarrow{G_2A}+\overrightarrow{G_2B}+\overrightarrow{G_2C} = \lambda\overrightarrow{G_2G}+\mu\overrightarrow{MH}$.
   (b) Deduce from it that G_2 is the center of mass of the points H, A, B, C.
   (c) What result have we shown ?

Exercise 6 *(3 pts)*

1. Draw the graph of the function : $f(x,y) = (\sqrt{1-x^2+y^2})$.
2. Let r, θ and z be the cylindrical coordinates.
   Consider the parametric curve :

$$\begin{cases} r(t) = 1 - \dfrac{1}{1+t} \\ \theta(t) = 2\pi t \\ z(t) = \dfrac{\sqrt{1+2t}}{1+t} \end{cases} \quad t \in \mathbb{R}^+$$

   (a) Express this parametric curve in cartesian coordinates, i. e. compute $x(t), y(t)$ and $z(t)$.
   (b) Show that this parametric curve is on the graph of f.
   (c) Let $M(t)$ be the point of coordinates $x(t), y(t)$ and $z(t)$. Write the vector $\overrightarrow{OM}(t)$ in cartesian coordinates and in the cartesian frame.
   (d) What is the velocity $\vec{v}$ of the point $M(t)$?
3. (a) Give the expression of the vectors $\vec{e_r}(t)$, $\vec{e_\theta}(t)$ and $\vec{e_z}(t)$ of the local cylindrical frame at the point $M(t)$.
   (b) Write the vector $\overrightarrow{OM}(t)$ in the cylindrical frame.
   (c) Deduce from it the velocity $\vec{v}$ of the point $M(t)$ in the cylindrical frame.
4. Give an approximate representation of the trajectory of $M(t)$.
