Physics: Exam n°1

Thursday 23 October 2014 — **Duration: 1h45**

***Approximate marking scheme**: Exercise 1, 11 points; Exercise 2, 9 points.*

The exam comprises two separate parts.

- Part 1 evaluates "assessment objective 1" ("develop a scientific approach to problem solving").
- Part 2 evaluates "objective 2" ("acquire and master enduring knowledge").

Document permitted in the exam: a single, two-sided, <u>personal,</u> hand-written Synopsis Sheet.

Authorised Calculator: non programmable, "college" model

<u>Exercise 1:</u> Study of a magnetic field with rotational symmetry

Formula list

Approximations: $f(z+dz) \approx f(z) + \frac{\partial f(z)}{\partial z} \cdot dz$ and $\frac{\partial f(z+dz)}{\partial z} \approx \frac{\partial f(z)}{\partial z} + \frac{\partial^2 f(z)}{\partial z^2} \cdot dz$

Maxwell's equations in vacuum:

$$\overrightarrow{rot}(\vec{B}) = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \quad avec \quad \vec{j} = \gamma \vec{E}, \quad div(\vec{B}) = 0, \quad \overrightarrow{\mathrm{rot}}(\vec{E}) = -\frac{\partial \vec{B}}{\partial t}, \quad div(\vec{E}) = \frac{\rho}{\varepsilon_0}$$

Using cylindrical coordinates: $\vec{X} = X_r \, \vec{u_r} + X_\theta \, \vec{u_\theta} + X_z \, \vec{u_z}$ and

$$\overrightarrow{rot}(\vec{X}) = \frac{1}{r}\left(\frac{\partial X_z}{\partial \theta} - \frac{\partial (rX_\theta)}{\partial z}\right)\vec{u_r} + \left(\frac{\partial X_r}{\partial z} - \frac{\partial X_z}{\partial r}\right)\vec{u_\theta} + \frac{1}{r}\left(\frac{\partial (rX_\theta)}{\partial r} - \frac{\partial X_r}{\partial \theta}\right)\vec{u_z}$$

Please Note: Part B uses the results of Part A which are in fact given in the exam paper. It can therefore be attempted even if Part A is not completed.

Consider a circular loop of radius R through which flows a current (Figure 1). Use cylindrical coordinates $(\vec{u_r}, \vec{u_\theta}, \vec{u_z})$.

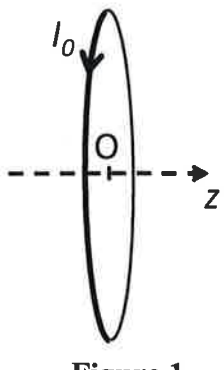

Figure 1

A) The current is constant and equal to I_0.

A.1) Give, <u>**and justify**</u>, the topography of the magnetic field $\vec{B}$.
What can one say about the component B_θ? Write the general form of field $\vec{B}$ associated with the loop. Specify the expression of $\vec{B}$ at $r = 0$ and show, on a sketch, the components of $\vec{B}$ for two points not situated on the axis Oz *but symmetrical with respect to Oz.*

A.2) *The component of field $\vec{B}$ along z for a point on the axis Oz can be written:*

$$B_z(r=0,z) = B_z(z) = B_0\left(1+\frac{z^2}{R^2}\right)^{-\frac{3}{2}} \quad where \quad B_0 = \frac{\mu_0 I_0}{2\pi R}.$$

<u>Without doing any calculations</u>, give a method of obtaining this expression, explaining it in a few lines. Comment on this expression.

Knowing the magnetic field on the axis, we will derive the field at a point off the axis but very close to it (at a very small distance r). To do that, we will exploit the properties of $\vec{B}$ (§A.3: properties of the flux of $\vec{B}$, §A.4: properties of the circulation of $\vec{B}$).

A.3) *We will first try to express the radial component* $B_r(r,z)$ *for points very close to the axis as a function of* $B_z(z)$*. To do so, we will establish an approximate expression of the flux of* $\vec{B}$ *through a closed surface around a small cylinder centered on Oz, of radius r and length dz (r and dz being infinitely small) (Figure 2).*

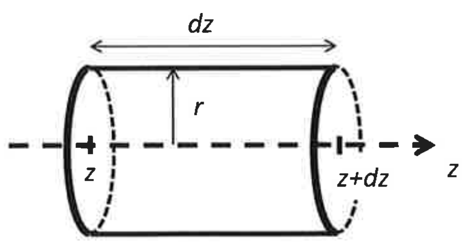

Figure 2

For this question, you will do following approximations:

$$B_z(r,z) \approx B_z(r=0,z) = B_z(z)$$

and $\quad B_z(r,z+dz) \approx B_z(r=0,z+dz) = B_z(z+dz)$.

Furthermore, we will consider that the radial component at the distance r from the axis is equal to $B_r(r,z)$ *all along the cylinder of length dz.*

What property of the magnetic field are you going to use? Show that you can then get the relation :

$$\boxed{B_r(r,z) = -\frac{r}{2}\cdot f(z) \qquad \text{with} \qquad f(z) = \left(\frac{\partial B_z(z)}{\partial z}\right)}.$$

A.4) *We will now express the axial component* $B_z(r,z)$ *for points very close to the axis as a function of* $B_z(z)$.

To do so, we will establish an approximate expression of the circulation of $\vec{B}$ *along an orientated rectangle MNOP (Figure 3).*

What property of the magnetic field are you going to use? Using the result from the previous question, show that you can then get the relation:

$$\boxed{B_z(r,z) = B_z(z) - \frac{r^2}{4}\cdot\frac{\partial^2 B_z(z)}{\partial z^2}}$$

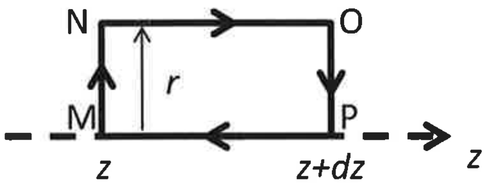

Figure 3

NB : In the rest of the subject, we will neglect the second term of the Taylor series for $B_z(r,z)$**.**

A.5) From the relation previously obtained in question §A.3 and for the magnetic field defined in §A.2, determine the expression of $B_r(r,z)$.

B) Now, the current in the loop *i(t)* varies with time, but slowly enough to consider that the topography and the expression of $\vec{B}$ do not change.

B.1) Considering Maxwell-Ampère equation and that the loop is in air (insulating medium of conductivity γ nil), show the existence of a $\vec{E}$ field collinear to $\overrightarrow{u_\theta}$. In a plane parallel to the loop situated at position $z = Z$, describe using a sketch the shape of the field lines of $\vec{E}$.

B.2) Starting with Stokes' theorem, establish that the electric field E satisfies the following integral-form equation: $\oint_{\Gamma^+} \vec{E}.\overrightarrow{d\ell} = -\iint_{(S)} \frac{\partial \vec{B}}{\partial t}.\overrightarrow{dS}$. Then calculate the electric field in the plane z=Z, at the distance $r = \rho$ to the axis Oz (ρ is small) in the particular case where the current $i(t)$ is of the form: $i(t) = I_m \cos(\omega t)$. Justify carefully all the steps of the calculation.

A wire-type loop of radius $r = \rho$*, made of a metallic wire of conductivity γ, is placed in this plane at the position* $z = Z$*, and its center is located on the axis Oz.* Describe qualitatively what occurs in the loop.

Exercise 2: Study of the electrostatic field and potential created by a charge distribution with spherical symmetry

We consider a distribution of electrostatic charges having a spherical symmetry. The charge is a volume charge and it is contained between 2 concentric spheres of center O and radii a and b (a < b). Its density is a function of the distance r to the center O according to: $\rho = k \cdot r$ *(k is a constant) for* $a < r < b$ *and it is nil elsewhere. We denote Q the total charge contained between the 2 spheres. We will use spherical coordinates. The position of a point M in space is given by:* $\overrightarrow{OM} = r \cdot \vec{u}_r$.

1) Express k as a function of Q, a and b for $a < r < b$.

2) Give and justify the topography of the electrostatic field $\vec{E}$.

3) Determine the electrostatic field $\vec{E}$ at M, in terms of the data presented in the exercise; Q, a, b, r, ε_0 and $\vec{u}_r$. We will distinguish 3 regions: $\vec{E}^+$ for $r > b$, $\vec{E}^i$ for $a < r < b$ and $\vec{E}^-$ for $0 < r < a$. What happens for $r = 0$?

4) Without doing the calculation, propose and describe a method to calculate the potential V in the 3 regions, including the determination of integration constants.

5) a) Sketch the functions $\rho(r)$ and $E(r)$.

   b) Assume that a gets very close to b, then the charge distribution becomes an infinitely thin layer of radius b. Calculate the corresponding surface charge σ.
   What can we say about the electric field E upon crossing the charged layer of surface charge σ?