INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES LYON **SCAN 2 — Math Test #1 — 1h30** *November 6, 2014*

No documents, no calculators, no cell phones or electronic devices allowed but you may keep your pet blobfish for moral support.

All your answers must be fully justified, unless noted otherwise.

Exercise 1.

1. Show that for all $A \in \mathbb{R}$, the following improper integral converges, and determine its value:

$$\int_A^{+\infty} \mathrm{e}^{-t}\,\mathrm{d}t.$$

2. Show that for all $x \in \mathbb{R}_+^*$, the following improper integral is convergent:

$$\int_x^{+\infty} \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t.$$

   Show that if $x = 0$, this improper integral is divergent.

   We define the function F as

$$F : \begin{array}{ccc} \mathbb{R}_+^* & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \displaystyle\int_x^{+\infty} \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t. \end{array}$$

3. Show that F is of class C^1 on $\mathbb{R}_+^*$ and determine an expression of F' on $\mathbb{R}_+^*$. *Hint: For $x \in \mathbb{R}_+^*$, express $F(x) - F(1)$ as a definite integral.*

4. a) Show that for all $x \in \mathbb{R}_+^*$,

$$0 \leq xF(x) \leq \int_x^{+\infty} \mathrm{e}^{-t}\,\mathrm{d}t.$$

   b) Deduce the value of the limit $\lim_{x \to +\infty} xF(x)$.

5. a) Show that for all $x \in (0, 1]$,

$$0 \leq F(x) \leq F(1) + \int_x^1 \frac{\mathrm{d}t}{t} = -\ln x.$$

   b) Deduce that $\lim_{x \to 0+} xF(x) = 0$.

6. Use an integration by parts to show that the following improper integral converges, and determine its value:

$$I = \int_0^{+\infty} F(x)\,\mathrm{d}x.$$

Exercise 2. Let $E = C([0,1])$ be the vector space that consists of all continuous real-valued functions on $[0,1]$. We define

$$\begin{aligned} N : E &\longrightarrow \mathbb{R} \\ f &\longmapsto \int_0^1 t|f(t)|\,\mathrm{d}t. \end{aligned}$$

1. Show that N is a norm on E.

2. Show that there exists $A \in \mathbb{R}_+^*$ (that you will determine) such that

$$\forall f \in E, \quad N(f) \leq A\|f\|_1.$$

3. For $n \in \mathbb{N}^*$ define the function f_n by

$$\begin{aligned} f_n : [0,1] &\longrightarrow \mathbb{R} \\ x &\longmapsto \begin{cases} n(1-nx) & \text{if } x \in [0,1/n] \\ 0 & \text{if } x \in [1/n,1]. \end{cases} \end{aligned}$$

   The function f_n is an element of E (you don't have to justify this fact).

   a) Compute, for $n \in \mathbb{N}$, the value of $N(f_n)$ and of $\|f_n\|_1$.

   b) Show that the sequence $(f_n)_{n\in\mathbb{N}^*}$ converges in (E, N) to an element $f \in E$ that you will determine.

   c) Show that the sequence $(f_n)_{n\in\mathbb{N}^*}$ doesn't converge to f in $(E, \|\cdot\|_1)$.

   d) Are the norms N and $\|\cdot\|_1$ equivalent?

4. More generally: let $g : [0,1] \longrightarrow \mathbb{R}_+$ be a continuous function. Define the mapping N_g as

$$\begin{aligned} N_g : E &\longrightarrow \mathbb{R} \\ f &\longmapsto \int_0^1 |f(t)|g(t)\,\mathrm{d}t. \end{aligned}$$

   Give, without any justifications, a necessary and sufficient condition on g for N_g to be a norm equivalent to $\|\cdot\|_1$ and, when this condition is fulfilled, explicit (without any justifications) coefficients $\alpha, \beta \in \mathbb{R}_+^*$ such that

$$\forall f \in E, \quad \alpha\|f\|_1 \leq N_g(f) \leq \beta\|f\|_1.$$

Exercise 3. Define the function f as

$$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (x^2+y^2, xy). \end{aligned}$$

Let $(x_0, y_0) \in \mathbb{R}^2$. Show (by using the definition) that f is differentiable at (x_0, y_0) and explicit $D_{(x_0,y_0)}f$.