SCAN 2 — Math Test #3 — 1h30 $\pi - 1$, *2014*

No documents, no calculators, no cell phones or electronic devices allowed.

All your answers must be fully justified, unless noted otherwise.

**Exercise 1.** In each of the following cases, compute (if it exists) the sum of the series $\sum_n u_n$, where

$$(1)\ \forall n \in \mathbb{N},\quad u_n = \frac{n^2}{n!},\qquad (2)\ \forall n \geq 2,\quad u_n = \ln\left(1 - \frac{1}{n^2}\right),\qquad (3)\ \forall n \in \mathbb{N},\quad u_n = \frac{n^n}{n!}.$$

**Exercise 2.**

1. Show that the function $f$ defined on $\mathbb{R}_+^*$ by

$$\forall x \in \mathbb{R}_+^*,\quad f(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{1+nx}$$

is (well-defined and) continuous on $\mathbb{R}_+^*$.

2. Let $a > 0$. Determine $N_a \in \mathbb{N}$ such that

$$\forall x \in [a, +\infty),\quad \left| f(x) - \sum_{n=0}^{N_a} \frac{(-1)^n}{1+nx} \right| < 10^{-3}.$$

**Exercise 3.** Define the sequences of functions $(f_n)_{n\in\mathbb{N}}$ and $(F_n)_{n\in\mathbb{N}}$ on $\mathbb{R}_+$ by:

$$\forall n \in \mathbb{N},\quad \forall x \in \mathbb{R}_+,\qquad f_n(x) = \frac{x^3}{\left(1+x^2\right)^n},\qquad F_n(x) = \int_0^x f_n(t)\,\mathrm{d}t.$$

1. Show that the sequence of functions $(F_n)_{n\in\mathbb{N}}$ converges pointwise on $\mathbb{R}_+$ to a function $F$ you will determine.
2. Does the sequence of functions $(F_n)_{n\in\mathbb{N}}$ converge uniformly to $F$ on $\mathbb{R}_+$?

**Exercise 4.** Define the sequence of functions $(u_n)_{n\in\mathbb{N}^*}$ on $\mathbb{R}_+$ by

$$\forall n \in \mathbb{N}^*,\quad \forall x \in \mathbb{R}_+,\qquad u_n(x) = \frac{x}{n(1+nx^2)}.$$

1. Show that the series of functions $\sum_n u_n$ converges uniformly on $\mathbb{R}_+$.
2. We denote by $f$ the sum of this series:

$$f = \sum_{n=1}^{+\infty} u_n.$$

   a) Show that $f$ is of class $C^1$ on $(0, +\infty)$.

   b) Use a comparison with an integral to show that

$$\forall x \in \mathbb{R}_+^*,\qquad x\ln\left(\frac{1+x^2}{x^2}\right) \leq f(x) \leq \frac{x}{1+x^2} + x\ln\left(\frac{1+x^2}{x^2}\right),$$

   and deduce that:

   i) $f$ is not differentiable at 0.

   ii) $\lim_{x\to 0^+} f(x) = 0$.

   iii) $\lim_{x\to+\infty} f(x) = 0$.