SCAN 2 — Math Test #2 — 1h30

December 12, 2013

No documents, no calculators, no cell phones or electronic devices allowed.

All your answers must be fully justified, unless noted otherwise.

Exercise 1. Determine, if it exists, the second-order Taylor–Young expansion of the function f defined by

$$f: \begin{array}{rcl} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & xe^{x+2y}. \end{array}$$

at the point $(1,2)$.

Exercise 2. The goal of this exercise is to determine all the real-valued functions f of class C^1 on $\mathbb{R}^2$ that satisfy the partial differential equation

$$\text{(E)} \qquad 2\partial_1 f(x,y) + \partial_2 f(x,y) + (2y-x)f(x,y) = 0$$

on $\mathbb{R}^2$.

1. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function of class C^1 and set

$$g: \begin{array}{rcl} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (u,v) & \longmapsto & f(2u+v, u+v). \end{array}$$

   Explain why g is of class C^1 on $\mathbb{R}^2$ and determine the first order partial derivatives of g in terms of that of f.

2. Show that f is a solution of Equation (E) on $\mathbb{R}^2$ if and only if g is a solution of

$$\text{(E}'\text{)} \qquad \partial_1 g(u,v) + vg(u,v) = 0$$

   on $\mathbb{R}^2$.

3. Find all the solutions g of class C^1 on $\mathbb{R}^2$ of Equation (E′).

4. Deduce the general solution of class C^1 of Equation (E) on $\mathbb{R}^2$.

Exercise 3. Let U be an open subset in normed vector space $(E, \|\cdot\|)$, let $a_0 \in U$ and let f_1, f_2 and g be three real-valued functions defined on U such that:

- f_1 and f_2 are differentiable at a, • $\forall x \in U, f_1(x) \leq f_2(x)$
- $f_1(a_0) = f_2(a_0)$,

We recall that if $h \in E$, the directional derivative of a function f at a_0 in the direction h is the value of the limit

$$\nabla_h f(a_0) = \lim_{t \to 0} \frac{f(a_0 + th) - f(a_0)}{t}$$

if it exists in $\mathbb{R}$. Moreover, we recall that if f is differentiable at a_0, then all the directional derivatives of f at a_0 exist and that for all $h \in E$, $\mathrm{d}_{a_0} f(h) = \nabla_h f(a_0)$.

1. Let $h \in E$.

   a) Show that there exists $\varepsilon > 0$ such that for all $t \in (-\varepsilon, \varepsilon)$, $a_0 + th \in U$.

   b) For $t \in (0, \varepsilon)$, compare the quantities:

   $$\frac{f_1(a_0 + th) - f_1(a_0)}{t} \quad \text{and} \quad \frac{f_2(a_0 + th) - f_2(a_0)}{t}.$$

   Deduce that $\nabla_h f_1(a_0) \leq \nabla_h f_2(a_0)$.

   c) Study the case $t \in (-\varepsilon, 0)$.

2. Deduce that $\mathrm{d}_{a_0} f_1 = \mathrm{d}_{a_0} f_2$.

3. Let g be a real-valued function on U such that:

   $$\forall x \in U, \quad f_1(x) \leq g(x) \leq f_2(x).$$

   Show that g is differentiable at a_0 and that $\mathrm{d}_{a_0} g = \mathrm{d}_{a_0} f_1$.

Exercise 4. We denote by $M_2(\mathbb{R})$ the real vector space of 2×2 matrices with real coefficients. Show that the determinant map

$$\begin{array}{rccc} \det : & M_2(\mathbb{R}) & \longrightarrow & \mathbb{R} \\ & A & \longmapsto & \det A \end{array}$$

is differentiable at $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and determine $\mathrm{d}_I \det$.