SCAN 2 — Math Test #1 — 1h30 *October 24, 2013*

No documents, no calculators, no cell phones or electronic devices allowed.

All your answers must be fully justified, unless noted otherwise.

**Exercise 1.** Solve the following linear differential system:

$$(S)\qquad \begin{cases} x'(t) = -2x(t) + y(t) + z(t) \\ y'(t) = x(t) - 2y(t) + z(t) \\ z'(t) = x(t) + y(t) - 2z(t). \end{cases}$$

Is the set of solutions of (S) a vector space (no justifications required)? if yes, give its dimension (with no justifications).

**Exercise 2.** Find the solution of

$$(*)\qquad xy'(x) + (x^2+1)y(x) = x^2 + 1$$

on $\mathbb{R}_+^*$ that satisfies $y(1) = \sqrt{e} + 1$.

**Exercise 3.**

1. Show that $N$ defined by

$$N:\ \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & |2x+y| + |x+y| \end{array}$$

is a norm on $\mathbb{R}^2$.

2. Plot the closed unit ball $\overline{B}$ of $N$.
3. Explain why the norms $N$ and $\|\cdot\|_1$ are equivalent.
4. Explicitly determine two real numbers $\alpha > 0$ and $\beta > 0$ such that $\alpha\|\cdot\|_1 \leq N \leq \beta\|\cdot\|_1$, and show on a graph the inclusions of balls this inequality represents. *A graphical determination of $\alpha$ and $\beta$ will be considered valid.*

**Exercise 4.** Let $E = C([0,1])$ be the real vector space of real valued continuous functions on $[0,1]$ and let $(f_n)_{n\in\mathbb{N}^*}$ be the sequence of elements of $E$ defined by:

$$\forall n \in \mathbb{N}^*, \quad \forall x \in [0,1], \quad f_n(x) = nxe^{-nx}.$$

1. On the same graph, sketch the graph of $f_n$ for three values of $n$. You will first (quickly and without details) determine the variations of $f_n$ on $[0,1]$.
2. For $x \in [0,1]$ show that the limit $\lim\limits_{n\to+\infty} f_n(x)$ exists in $\mathbb{R}$ and determine its value, that will be denoted by $f(x)$. Is $f$ an element of $E$?
3. For $n \in \mathbb{N}^*$, determine $\|f_n\|_\infty$ and $\|f_n\|_1$.
4. a) From the graphs of the $f_n$'s, what can you guess about the convergence of $(f_n)_{n\in\mathbb{N}^*}$ to $f$ in $(E, \|\cdot\|_\infty)$?
   b) Prove your previous statement.
5. a) From the graphs of the $f_n$'s, what can you guess about the convergence of $(f_n)_{n\in\mathbb{N}^*}$ to $f$ in $(E, \|\cdot\|_1)$?
   b) Prove your previous statement.
6. Are the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ equivalent norms on $E$?