SCAN 1 — Solution of Math Test #4

Romaric Pujol, romaric.pujol@insa-lyon.fr

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Exercise 1.

1. Let $N \in \mathbb{N}$ and $f : [a,b] \longrightarrow \mathbb{R}$ be a function of class C^N on $[a,b]$ and $N+1$ times differentiable on (a,b). Then there exists $c \in (a,b)$ such that
$$f(b) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(N+1)}(c)}{(N+1)!}(b-a)^{N+1}.$$

2. The function exp is of class C^4 on $[0,1/2]$ and five times differentiable on $(0,1/2)$ (it's even of class C^∞ on $\mathbb{R}$) hence there exists $c \in (0,1/2)$ such that
$$\begin{aligned}\sqrt{e} = \exp(1/2) &= \sum_{n=0}^{4} \frac{\exp^{(n)}(0)}{n!}\left(\frac{1}{2}\right)^n + \frac{\exp^5(c)}{5!}\left(\frac{1}{2}\right)^5 \\ &= 1 + \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{6}\left(\frac{1}{2}\right)^3 + \frac{1}{24}\left(\frac{1}{2}\right)^4 + \frac{e^c}{120}\left(\frac{1}{2}\right)^5 \\ &= \frac{24\times 16 + 24\times 8 + 24\times 2 + 8 + 1}{24\times 16} + \frac{e^c}{120}\frac{1}{32} \\ &= \frac{24\times 26+9}{24\times 16} + \frac{e^c}{120}\frac{1}{32} \\ &= \frac{8\times 26+3}{8\times 16} + \frac{e^c}{120}\frac{1}{32} \\ &= \frac{211}{128} + \frac{e^c}{120}\frac{1}{32}.\end{aligned}$$
Now, since $c \in (0,1/2)$ and since $0 \le e \le 4$ we have $0 \le e^c \le 2$, hence
$$\frac{211}{128} \le \sqrt{e} \le \frac{211}{128} + \frac{2}{120}\frac{1}{32} = \frac{211}{128} + \frac{1}{1920}.$$

3. Hence,
$$\underline{1.6484} \le \frac{211}{128} \le \sqrt{e} \le \frac{211}{128} + \frac{1}{1920} \le \underline{1.648}96.$$
Hence the value of $\sqrt{e}$ correct to three to three decimal places is 1.648.

Exercise 2.

1. For all $x \in \mathbb{R}$,
$$\begin{aligned}\frac{1}{x^2+x+1} &= \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} \\ &= \frac{1}{\frac{3}{4}\left(\frac{4}{3}\left(x+\frac{1}{2}\right)^2+1\right)} \\ &= \frac{4}{3}\frac{1}{\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right)^2+1}.\end{aligned}$$
Hence,
$$\begin{aligned}\int \frac{dx}{x^2+x+1} &= \frac{4}{3}\int \frac{dx}{\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)^2+1} \\ &= \frac{4}{3}\frac{\sqrt{3}}{2}\arctan\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right) + C \\ &= \frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right) + C, \quad C \in \mathbb{R}.\end{aligned}$$

2. By a straightforward long division, we obtain:
$$\frac{2x^3+x^2}{x^3-1} = 2 + \frac{x^2+2}{x^3-1}.$$
Now, the polynomial function $x \mapsto x^3-1$ can be factored in $\mathbb{R}$ as $x \mapsto (x-1)(x^2+x+1)$; the partial fraction decomposition of $(x^2+2)(x^3-1)$ has the form:
$$\frac{x^2+2}{x^3-1} = \frac{x^2+2}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}.$$
To determine A we multiply this equality by $x-1$ and evaluate at $x=1$, and we obtain:
$$\frac{3}{3} = A$$
hence $A=1$. To determine B we multiply by x and take the limit as $x \to +\infty$, and we obtain:
$$1 = A + B$$
hence $B = 0$. To determine C we evaluate the equality at $x=0$ and we obtain:
$$-2 = -A + C$$
hence $C = -1$.

3. From the previous decomposition and the result of Question 1 we obtain:
$$\begin{aligned}\int \frac{2x^3+x^2}{x^3-1}\,dx &= \int\left(2 + \frac{1}{x-1} - \frac{1}{x^2+x+1}\right)dx \\ &= 2x + \ln|x-1| - \frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right) + C, \quad C \in \mathbb{R},\end{aligned}$$
and this formula is valid on $(-\infty,1)$ and on $(1,+\infty)$.

Exercise 3.

1. Since $2x - x^2 \underset{x\to 0}{\longrightarrow} 0$, we obtain from the usual Taylor–Young expansion of exp at 0:
$$\begin{aligned}e^{2x-x^2} &\underset{x\to 0}{=} 1 + (2x-x^2) + \frac{(2x-x^2)^2}{2} + \frac{(2x-x^2)^3}{6} + \frac{(2x-x^2)^4}{24} + o\left((2x-x^2)^4\right) \\ &\underset{x\to 0}{=} 1 + (2x-x^2) + \frac{1}{2}(4x^2-4x^3+x^4) + \frac{1}{6}(8x^3-12x^4) + \frac{1}{24}16x^4 + o(x^4) \\ &\underset{x\to 0}{=} 1 + 2x + x^2 - \frac{2}{3}x^3 - \frac{5}{6}x^4 + o(x^4).\end{aligned}$$
Also, from the usual Taylor–Young expansion of sin at 0:
$$4\sin(x) \underset{x\to 0}{=} 4x - \frac{2x^3}{3} + o(x^4)$$
we obtain:
$$f(x) \underset{x\to 0}{=} 1 - 2x + x^2 - \frac{5}{6}x^4 + o(x^4).$$

2. We have:
$$f(x) \underset{x\to 0}{=} 1 - 2x + x^2(1+o(x))$$
hence $f(x) - (1-2x) \underset{x\to 0}{=} x^2(1+o(x)) \ge 0$, hence, in the neighborhood of 0, the graph of f lies above the straight line of equation $y = 1-2x$. Also,
$$f(x) - (1-2x+x^2) \underset{x\to 0}{=} x^4\left(-\frac{5}{6} + o(1)\right)$$
hence, in the neighborhood of 0, the graph of f lies below the parabola of equation $y = 1-2x+x^2$. The graph of f is sketched on Figure 2.

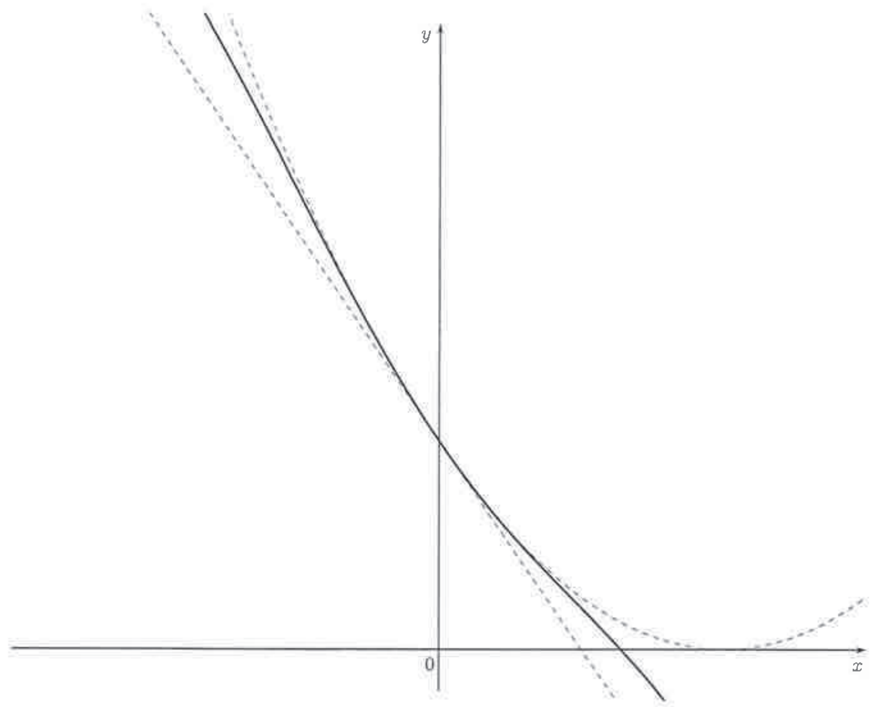

Figure 2: Graph of the function f of Exercise 3 where we have also represented (in dashed strokes) the curves $y = 1 - 2x$ and $y = 1 - 2x + x^2$

Exercise 4.

1. Let $x \in \mathbb{R}$. Then:

a) The case $f(x) = 1$:

$$(f * \exp)(x) = \int_0^x f(t)e^{x-t}\,dt = \int_0^x e^{x-t}\,dt = \left[-e^{x-t}\right]_{t=0}^{t=x} = -1 + e^x,$$

$$(\exp * f)(x) = \int_0^x e^t f(x-t)\,dt = \int_0^x e^t\,dt = \left[e^t\right]_{t=0}^{t=x} = e^x - 1.$$

b) The case $f(x) = x$:

$$\begin{aligned}(f * \exp)(x) &= \int_0^x f(t)e^{x-t}\,dt = \int_0^x te^{x-t}\,dt = e^x \int_0^x te^{-t}\,dt \\ &= e^x\left(\left[-te^{-t}\right]_{t=0}^{t=x} + \int_0^x e^{-t}\,dt\right), \qquad \textit{by an integration by parts} \\ &= e^x\left(-xe^{-x} - e^{-x} + 1\right) \\ &= -x - 1 + e^x, \\ (\exp * f)(x) &= \int_0^x e^t f(x-t)\,dt = \int_0^x e^t(x-t)\,dt = x\int_0^x e^t - \int_0^x te^t\,dt \\ &= x(e^x - 1) - \left(\left[te^t\right]_{t=0}^{t=x} - \int_0^x e^t\,dt\right), \qquad \textit{by an integration by parts} \\ &= xe^x - x - (xe^x - e^x + 1) \\ &= -x - 1 + e^x.\end{aligned}$$

2. Let $x \in \mathbb{R}$. In the following integral, we use the substitution $s = x - t$, so that $dt = -ds$:

$$\begin{aligned}(f * g)(x) &= \int_0^x f(t)g(x-t)\,dt \\ &= -\int_x^0 f(x-s)g(s)\,ds \\ &= \int_0^x g(s)f(x-s)\,ds \\ &= (g * f)(x).\end{aligned}$$

3. Let $x \in \mathbb{R}$. Then:

$$(f * \exp)(x) = \int_0^x f(t)e^{x-t}\,dt = e^x \int_0^x f(t)e^{-t}\,dt.$$

Since the function $t \mapsto f(t)e^{-t}$ is continuous on $\mathbb{R}$ (as f is continuous on $\mathbb{R}$), the function $x \mapsto \int_0^x f(t)e^{-t}\,dt$ is differentiable on $\mathbb{R}$ and for all $x \in \mathbb{R}$,

$$\frac{d}{dx}\int_0^x f(t)e^{-t}\,dt = f(x)e^{-x}.$$

Hence $(f * \exp)$ is a product of differentiable functions on $\mathbb{R}$, and for all $x \in \mathbb{R}$ one has:

$$(f * \exp)'(x) = e^x \int_0^x f(t)e^{-t} + e^x\left(f(x)e^{-x}\right)$$

(by the product rule), i.e.,

$$(f * \exp)'(x) = \int_0^x f(t)e^{x-t}\,dt + f(x) = (f * \exp)(x) + f(x)$$

hence the result. Now if f is differentiable on $\mathbb{R}$,

$$(f * \exp)'' = \left((f * \exp)\right)' = \left((f * \exp) + f\right)' = (f * \exp)' + f' = (f * \exp) + f + f'.$$

4. Let f be a differentiable function on $\mathbb{R}$. Then:

$$\begin{aligned}(f * \exp) \text{ is a solution of (E)} &\iff (f * \exp)'' - 2(f * \exp)' + (f * \exp) = h \\ &\iff (f * \exp) + f + f' - 2\left((f * \exp) + f\right) + (f * \exp) = h \\ &\iff f' - f = h.\end{aligned}$$

5. A particular solution of the differential equation

$$f'(x) - f(x) = -x + 1$$

is clearly given by $f_1(x) = x$. Hence a particular solution of (E) is $(f_1 * \exp)$, and as we already computed:

$$\forall x \in \mathbb{R}, \quad (f_1 * \exp)(x) = -x - 1 + e^x.$$

Now, the homogeneous differential equation associated with (E) is:

(H) $$y'' - 2y' + y = 0.$$

Its characteristic equation is

$$r^2 - 2r + 1 = 0$$

with discriminant 0 and unique solution 1, hence the general solution of (H) is:

$$y(x) = (Ax + B)e^x, \qquad A, B \in \mathbb{R}.$$

Hence the general solution of equation (E) on $\mathbb{R}$ is:

$$f(x) = (Ax + B)e^x - x - 1 + e^x, \quad A, B \in \mathbb{R}$$

that can be written as

$$f(x) = (Ax + B)e^x - x - 1, \quad A, B \in \mathbb{R}.$$