SCAN 1 — Solution of Math Test #3

Romaric Pujol, romaric.pujol@insa-lyon.fr

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Exercise 1.

1. a) Since $\cos x \underset{x\to 0}{\longrightarrow} 1$, and since $\cos x - 1 \underset{x\to 0}{\sim} -x^2/2$,

$$\ln(\cos x) \underset{x\to 0}{\sim} \cos x - 1 \underset{x\to 0}{\sim} -\frac{x^2}{2}.$$

Hence

$$\frac{\ln(\cos x)}{x} \underset{x\to 0}{\sim} -\frac{x}{2}.$$

b) For all $x \in (-\pi/2, 0) \cup (-\pi/2, 0)$ (which is a punctured neighborhood of 0),

$$(\cos x)^{1/x} - 1 = \exp\left(\frac{\ln(\cos x)}{x}\right) - 1.$$

Now, from the previous question

$$\lim_{x\to 0} \frac{\ln(\cos x)}{x} = \lim_{x\to 0} -\frac{x}{2} = 0$$

hence

$$(\cos x)^{1/x} - 1 \underset{x\to 0}{\sim} \frac{\ln(\cos x)}{x} \underset{x\to 0}{\sim} -\frac{x}{2}.$$

2. For all $x \in (0, +\infty)$ (which is a neighborhood of $+\infty$),

$$x^{1/x} - 1 = \exp\left(\frac{\ln x}{x}\right) - 1.$$

Now,

$$\lim_{x\to+\infty} \frac{\ln x}{x} = 0$$

hence

$$x^{1/x} - 1 \underset{x\to+\infty}{\sim} \frac{\ln x}{x}.$$

Exercise 2.

1. The function f is differentiable on $[-1, +\infty)$ and for all $x \in [-1, +\infty)$,

$$f'(x) = (x+1)e^x.$$

Hence, for all $x \in (-1, +\infty)$, $f'(x) > 0$, hence f is increasing on $[-1, +\infty)$. Hence f is $1-1$.

2. Since f is continuous and increasing, and since $[-1, +\infty)$ is an interval, J is the interval given by:

$$J = f([-1, +\infty)) = \left[f(-1), \lim_{x\to+\infty} f(x)\right).$$

Now, $\lim_{x\to+\infty} f(x) = +\infty$, hence $J = [-1/e, +\infty)$.

3. The function f is differentiable on $[-1, +\infty)$, and for all $x \in (-1, +\infty)$, $f'(x) \neq 0$. Hence W is differentiable on $f\big((-1, +\infty)\big) = (-1/e, +\infty)$. Since $f'(-1) = 0$, W is not differentiable at $f(-1) = -1/e$.

4. For all $y \in J = [-1/e, +\infty)$, $f\big(W(y)\big) = y$, hence for all $y \in \mathbb{R}_+^*$,

$$\ln y = \ln f\big(W(y)\big) = \ln\big(W(y)e^{W(y)}\big) = \ln W(y) + \ln e^{W(y)} = \ln W(y) + W(y).$$

Now, since $\lim_{y\to+\infty} W(y) = +\infty$,

$$\lim_{y\to+\infty} \frac{\ln W(y)}{W(y)} = 0$$

hence

$$W(y) \underset{y\to+\infty}{\sim} \ln y.$$

5. a)

$$(X-1)e^X = W(y)e^{W(y)-1} = W(y)e^{W(y)}e = f\big(W(y)\big)e = ye.$$

b) Since $\lim_{y\to(-1/e)^+} W(y) = -1$, $\lim_{y\to(-1/e)^+} X = 0$ hence

$$ey = (X-1)e^X \underset{y\to(-1/e)^+}{=} (X-1)\left(1 + X + \frac{X^2}{2} + o(X^2)\right) \underset{y\to(-1/e)^+}{=} -1 + \frac{X^2}{2} + o(X^2),$$

hence

$$X^2 \underset{y\to(-1/e)^+}{=} 2(ey+1) + o(X^2).$$

c) Hence

$$X^2 \underset{y\to(-1/e)^+}{\sim} 2(ey+1),$$

i.e.,

$$\big(W(y)+1\big)^2 \underset{y\to(-1/e)^+}{\sim} 2(ey+1).$$

Now, since for all $y \in (-1/e, +\infty)$, $W(y) + 1 > 0$,

$$W(y) + 1 \underset{y\to(-1/e)^+}{\sim} \sqrt{2(ey+1)}.$$

Exercise 3.

1. The function sinh is differentiable on $\mathbb{R}$ and the function arctan is differentiable on $\sinh(\mathbb{R}) = \mathbb{R}$, hence the function f is differentiable on $\mathbb{R}$. Moreover, for all $x \in \mathbb{R}$,

$$f'(x) = \sinh'(x)\arctan'(\sinh x) = \frac{\cosh x}{1 + \sinh^2 x} = \frac{\cosh x}{\cosh^2 x} = \frac{1}{\cosh x}.$$

2. Let $x \in \mathbb{R}_+^*$. Since the function f is continuous on $[0, x]$ and differentiable on $(0, x)$, by the Mean Value Theorem, there exists $c \in (0, x)$ such that

$$f(x) - f(0) = (x - 0)f'(c)$$

i.e.,

$$f(x) = \frac{x}{\cosh c}.$$

Now, since $c \in (0, x) \subset \mathbb{R}_+^*$ and since the function $t \mapsto 1/\cosh(t)$ is decreasing on $[0, x]$,

$$\frac{1}{\cosh x} < \frac{1}{\cosh c} < \frac{1}{\cosh 0} = 1$$

hence (since $x > 0$),

$$\frac{x}{\cosh x} < f(x) < x.$$

3. a) The function f is the composition of two increasing functions on $\mathbb{R}$, hence it is increasing on $\mathbb{R}$. Now, $f(0) = 0$, hence for all $x > 0$, $f(x) > 0$. An easy induction shows that (since $u_0 > 0$), for all $n \in \mathbb{N}$, $u_n \in \mathbb{R}_+^*$.

b) By Question 2, since for all $n \in \mathbb{N}$, $u_n \in \mathbb{R}_+^*$,

$$u_{n+1} = f(u_n) < u_n.$$

Hence the sequence $(u_n)_{n\in\mathbb{N}}$ is decreasing.

c) The sequence $(u_n)_{n\in\mathbb{N}}$ is decreasing and bounded from below (e.g., 0 is a lower bound), hence it is convergent.

d) Denote by ℓ the limit of $(u_n)_{n\in\mathbb{N}}$. Since 0 is a lower bound of $(u_n)_{n\in\mathbb{N}}$ we must have $\ell \geq 0$. We show that $\ell = 0$ by contradiction: assume that $\ell > 0$. Then, since f is continuous on $\mathbb{R}$,

$$\ell = \lim_{n\to+\infty} u_n = \lim_{n\to+\infty} u_{n+1} = \lim_{n\to+\infty} f(u_n) = f(\ell).$$

hence $\ell = f(\ell)$. By Question 2 again, we obtain:

$$\ell = f(\ell) < \ell$$

which is impossible. Hence $\ell = 0$.

4. Let $n \in \mathbb{N}$. Then:

$$\frac{u_{n+1} - u_n}{u_n^2} = \frac{1}{u_n}\left(\frac{f(u_n)}{u_n} - 1\right).$$

Now, since $u_n > 0$, we obtain, from Question 2:

$$\frac{1}{u_n}\left(\frac{1}{\cosh u_n} - 1\right) < \frac{u_{n+1} - u_n}{u_n^2} < 0,$$

hence

$$\frac{1 - \cosh u_n}{u_n \cosh u_n} < \frac{u_{n+1} - u_n}{u_n^2} < 0.$$

Now, since $\lim_{n \to +\infty} u_n = 0$,

$$\frac{1 - \cosh u_n}{u_n \cosh u_n} \underset{n \to +\infty}{\sim} \frac{-u_n^2/2}{u_n} = -\frac{u_n}{2} \underset{n \to +\infty}{=} 0.$$

Hence, by the Squeeze Theorem,

$$\lim_{n \to +\infty} \frac{u_{n+1} - u_n}{u_n^2} = 0,$$

hence

$$u_{n+1} - u_n \underset{n \to +\infty}{=} o(u_n^2),$$

hence the result.