SCAN 1 — Solution of Math Test #1

Romaric Pujol, romaric.pujol@insa-lyon.fr

October 18, 2013

Exercise 1.

1. $\forall x \in \mathbb{R},\ f(x) = 4(x-1)(x-2)$.
2. $\sin^2(\theta) = 1 - \cos^2(\theta)$ and $\cos(2\theta) = 2\cos^2(\theta) - 1$.
3. We first solve the equation in $\mathbb{R}$. We'll first transform the equation, using Question 2:

$$\begin{aligned}
\cos\left(\frac{2x}{3}\right) - 2\sin^2\left(\frac{x}{3}\right) - 12\cos\left(\frac{x}{3}\right) + 11 = 0 &\iff 2\cos^2\left(\frac{x}{3}\right) - 1 - 2 + 2\cos^2\left(\frac{x}{3}\right) - 12\cos\left(\frac{x}{3}\right) + 11 = 0\\
&\iff 4\cos^2\left(\frac{x}{3}\right) - 12\cos\left(\frac{x}{3}\right) + 8 = 0\\
&\iff \cos\left(\frac{x}{3}\right) = 1 \quad\text{or}\quad \cos\left(\frac{x}{3}\right) = 2 \qquad \textit{from Question 1}\\
&\iff \cos\left(\frac{x}{3}\right) = 1 \qquad \textit{since } \cos\left(\frac{x}{3}\right) = 2 \textit{ is impossible}\\
&\iff \exists k \in \mathbb{Z},\quad \frac{x}{3} = 2k\pi\\
&\iff \exists k \in \mathbb{Z},\quad x = 6k\pi.
\end{aligned}$$

We now select the solutions that lie in $[-4\pi, 8\pi]$: let $k \in \mathbb{Z}$, then:

$$\begin{aligned}
6k\pi \in [-4\pi, 8\pi] &\iff -4\pi \le 6k\pi \le 8\pi\\
&\iff -\frac{2}{3} \le k \le \frac{4}{3}\\
&\iff k \in \{0, 1\} \qquad \text{since } k \in \mathbb{Z}.
\end{aligned}$$

Hence the solutions of our equation that lie in $[-4\pi, 8\pi]$ are 0 and 6π.

Exercise 2.

- If $n = 0$ then $A = B$.
- If $n \ne 0$: since $B \ne 0$, and $A > 0$ and $B > 0$ we can compute $\frac{A}{B}$ and compare it with 1:

$$\begin{aligned}
\frac{A}{B} &= \frac{n! \times (1 \times 2 \times \cdots \times n)}{n! \times \big((n+1) \times (n+2) \times \cdots \times (2n)\big)}\\
&= \frac{1}{(n+1)} \times \frac{2}{n+2} \times \cdots \times \frac{n}{2n}.
\end{aligned}$$

Since each term in this factor is less than 1, we obtain: $\frac{A}{B} < 1$, i.e., $A < B$.

Exercise 3. For $n \in \mathbb{N}$ consider the property (P_n) defined as:

$$(P_n) \qquad \sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Let's prove by induction that for all $n \in \mathbb{N}$, (P_n) is true:

- The left-hand side of Property (P_0) is:

$$\sum_{k=0}^{0} k^2 = 0^2 = 0$$

and the right-hand side of Property (P_0) is also clearly 0, hence (P_0) is true.

- Assume that (P_n) is true for some $n \in \mathbb{N}$ and let's prove that (P_{n+1}) is true:

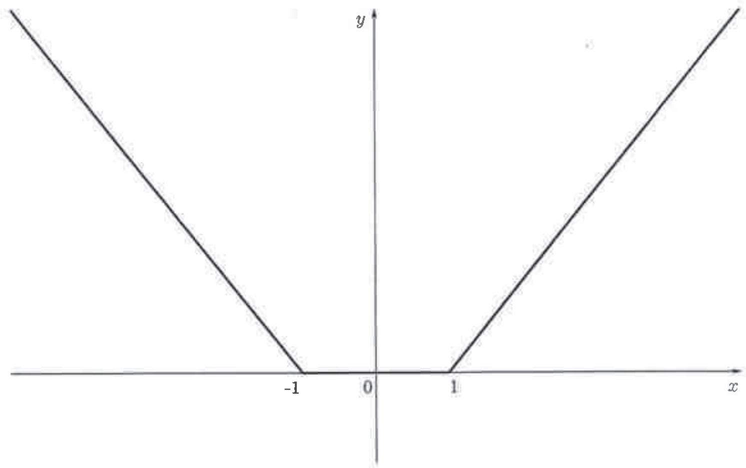
$$\begin{aligned}
\sum_{k=0}^{n+1} k^2 &= \left(\sum_{k=0}^{n} k^2\right) + (n+1)^2\\
&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \qquad \textit{since } (P_n) \textit{ is true}\\
&= \frac{1}{6}\Big(n(n+1)(2n+1) + 6(n+1)^2\Big)\\
&= \frac{1}{6}(n+1)\Big(n(2n+1) + 6(n+1)\Big)\\
&= \frac{1}{6}(n+1)\Big(2n^2 + n + 7n + 1\Big)\\
&= \frac{1}{6}(n+1)\Big(2n^2 + 6n + 6\Big)\\
&= \frac{1}{6}(n+1)(n+2)(2n+3)\\
&= \frac{(n+1)\big((n+1)+1\big)\big(2(n+1)+1\big)}{6}.
\end{aligned}$$

Hence (P_{n+1}) is true.

Hence, by induction, for all $n \in \mathbb{N}$, (P_n) is true.

Exercise 4.

1.



2. Let $a, b \in [-1, +\infty)$ such that $a < b$. There are three cases:
   - If $a < b \le 1$ then $f(a) = 0$ and $f(b) = 0$, hence $f(a) \le f(b)$,
   - If $a \le 1 \le b$ then $f(a) = 0$ and $f(b) = b - 1 \ge 0$ hence $f(a) \le f(b)$,
   - If $1 \le a < b$ then $f(a) = a - 1$ and $f(b) = b - 1$ and we indeed have $f(a) = a - 1 \le f(b) = b - 1$.

   Hence f is non-decreasing on $[-1, +\infty)$.
3. We read directly on the graph of f:

$$f\big((0,2]\big) = [0,1], \quad f\big([-1,1]\big) = \{0\}, \quad f(\mathbb{R}) = \mathbb{R}_+, \quad f\big(\{-1,1\}\big) = \{0\},$$
$$f^{[-1]}(\mathbb{R}_+^*) = (-\infty,-1) \cup (1,+\infty), \quad f^{[-1]}(\mathbb{R}_-^*) = \emptyset, \quad f^{[-1]}\big(\{0\}\big) = [-1,1], \quad f^{[-1]}\big([1,2]\big) = [-3,-2] \cup [2,3].$$

Exercise 5.

1.

$$\begin{aligned}
C &= \sum_{k=0}^{n} \alpha_k (x_k - A)^2 \\
&= \sum_{k=0}^{n} \alpha_k (x_k^2 - 2Ax_k + A^2) \\
&= \left(\sum_{k=0}^{n} \alpha_k x_k^2\right) - 2\left(A \sum_{k=0}^{n} \alpha_k x_k\right) + \left(A^2 \sum_{k=0}^{n} \alpha_k\right) \\
&= B - 2A^2 + A \qquad \textit{since } \sum_{k=0}^{n} \alpha_k = 1 \\
&= B - A^2.
\end{aligned}$$

2. a)

$$\begin{aligned}
\alpha_0 + \alpha_1 + \cdots + \alpha_n &= \sum_{k=0}^{n} \alpha_k \\
&= \sum_{k=0}^{n} 2^{-n} \binom{n}{k} \\
&= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^n \binom{n}{k} \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} \\
&= \left(\frac{1}{2} + \frac{1}{2}\right)^n \qquad \textit{by the Binomial Theorem} \\
&= 1^n = 1.
\end{aligned}$$

b)

$$\begin{aligned}
A &= \sum_{k=0}^{n} k 2^{-n} \binom{n}{k} \\
&= 2^{-n} \sum_{k=0}^{n} k \binom{n}{k} \\
&= 2^{-n} n 2^{n-1} \qquad \textit{by the given result} \\
&= \frac{n}{2}. \\
B &= \sum_{k=0}^{n} k^2 2^{-n} \binom{n}{k} \\
&= 2^{-n} \sum_{k=0}^{n} k^2 \binom{n}{k} \\
&= 2^{-n} \sum_{k=0}^{n} k(k-1+1) \binom{n}{k} \\
&= 2^{-n} \left(\sum_{k=0}^{n} k(k-1) \binom{n}{k} + \sum_{k=0}^{n} k \binom{n}{k}\right) \\
&= 2^{-n} \sum_{k=0}^{n} k(k-1) \binom{n}{k} + A \\
&= 2^{-n} n(n-1) 2^{n-2} + \frac{n}{2} \qquad \textit{by the given result} \\
&= \frac{n(n-1)}{4} + \frac{n}{2} \\
&= \frac{n(n+1)}{4}. \\
C &= B - A^2 \\
&= \frac{n(n+1)}{4} - \frac{n^2}{4} \\
&= \frac{n}{4}.
\end{aligned}$$

Exercise 6. Let's prove that f is $1-1$: let $a, b \in A$ such that $f(a) = f(b)$ and let's prove that $a = b$: if $a < b$ then $f(a) < f(b)$ (since f is increasing), but this is impossible since $f(a) = f(b)$; hence $a \geq b$. if $a > b$ then $f(a) > f(b)$ (since f is increasing), but this is impossible since $f(a) = f(b)$; hence $a = b$.