SCAN 1 — Math Test #4 — 2h00

$\pi + 7$, 2014

No documents, no calculators, no cell phones, no raccoons, no ponies allowed but you may keep your pet hedgehog for moral support.

All your answers must be fully justified, unless noted otherwise.

Exercise 1. The goal of this exercise is to compute an approximation of $\sqrt{\mathrm{e}}$ correct to three decimal places. You may use, without any justifications, the fact that $\mathrm{e} \in [0, 4]$.

1. Recall the N-th order Taylor–Lagrange formula for a function f on an interval $[a, b]$ (with $a < b$). Specify all the necessary hypotheses on f.
2. Show, applying the 4th order Taylor–Lagrange formula to the function exp, that

$$\frac{211}{128} \leq \sqrt{\mathrm{e}} \leq \frac{211}{128} + \frac{1}{1920}.$$

3. Using a calculator, we obtain:

$$\frac{211}{128} = 1.6484375, \qquad \cancel{\frac{79}{48} + \frac{1}{192} =} \; 1.648958\overline{3}, = \frac{211}{128} + \frac{1}{1920}$$

where the overlined digit is infinitely repeated. Deduce an approximation of $\sqrt{\mathrm{e}}$ correct to three decimal places. Note: your answer to this question must be well justified!

Exercise 2.

1. Compute an antiderivative of the function f defined on $\mathbb{R}$ by

$$\forall x \in \mathbb{R}, \quad f(x) = \frac{1}{x^2 + x + 1}.$$

Hint: complete the square.

2. Give the partial fraction decomposition of the following rational function:

$$\frac{2x^3 + x^2}{x^3 - 1}.$$

3. Deduce the form of the following indefinite integral

$$\int \frac{2x^3 + x^2}{x^3 - 1} \, \mathrm{d}x,$$

and specify the domain of validity.

Exercise 3. We define the function

$$\begin{aligned} f \,:\, \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \mathrm{e}^{2x - x^2} - 4\sin(x). \end{aligned}$$

1. Show that f admits a fourth-order Taylor–Young expansion at 0 of the form

$$f(x) \underset{x \to 0}{=} a + bx + cx^2 + dx^4 + o(x^4).$$

2. On the same figure, plot the curves $y = a + bx$, $y = a + bx + cx^2$, and sketch the graph of f in the neighborhood of 0. Explain.

Exercise 4 (*Product of Convolution*). If f and g are two continuous functions on $\mathbb{R}$ we define the *product of convolution of f and g* as the function denoted by $(f * g)$ and defined by

$$\forall x \in \mathbb{R}, \quad (f * g)(x) = \int_0^x f(t)g(x-t)\,\mathrm{d}t.$$

1. Give an explicit expression of the functions $(f * \exp)$ and $(\exp * f)$ in the following cases:

   a) The function f is defined by: $\forall x \in \mathbb{R}$, $f(x) = 1$;

   b) The function f is defined by: $\forall x \in \mathbb{R}$, $f(x) = x$.

2. Show, using a suitable substitution, that for all continuous functions f and g, $(f * g) = (g * f)$.

3. Let f be a continuous function on $\mathbb{R}$. Show that $(f * \exp)$ is differentiable on $\mathbb{R}$ and that $(f * \exp)' = (f * \exp) + f$. In the case when f is differentiable on $\mathbb{R}$, deduce an expression for $(f * \exp)''$.

4. Let h be a continuous function on $\mathbb{R}$ and consider the following differential equation:

   $$\text{(E)} \qquad y''(x) - 2y'(x) + y(x) = h(x).$$

   a) Let f be a differentiable function on $\mathbb{R}$. Show that $(f * \exp)$ is a solution of Equation (E) on $\mathbb{R}$ if and only if f is a solution of the differential equation

   $$f' - f = h$$

   on $\mathbb{R}$.

   b) In the case where $\forall x \in \mathbb{R}$, $h(x) = -x + 1$, deduce the general solution of Equation (E) on $\mathbb{R}$.