SCAN 1 — Math Test #3 — 1h30

January 10, 2014

No documents, no calculators, no cell phones, no raccoons allowed but you may keep your pet pony for moral support.

All your answers must be fully justified, unless noted otherwise.

Exercise 1. The two questions of this exercise are independent from each other.

1. a) Determine a simple equivalent of

$$\frac{\ln(\cos x)}{x}$$

as $x \to 0$.

b) Deduce a simple equivalent of $(\cos x)^{1/x} - 1$ as $x \to 0$.

2. Determine a simple equivalent of $x^{1/x} - 1$ as $x \to +\infty$.

Exercise 2 (*Lambert W function*). Let

$$\begin{aligned} f \,:\, [-1,+\infty) &\longrightarrow \mathbb{R} \\ x &\longmapsto x\mathrm{e}^x. \end{aligned}$$

1. Show that f is $1-1$.
2. Determine the range J of f.

We now consider f from $[-1,+\infty)$ to J so that f is a bijection, and we denote by W its inverse:

$$W = f^{-1} : J \longrightarrow [-1,+\infty).$$

3. Where is the function W differentiable? where is the function W *not* differentiable?
4. Show that

$$\forall y \in \mathbb{R}_+^*, \quad \ln\bigl(W(y)\bigr) + W(y) = \ln y,$$

and that $W(y) \underset{y\to+\infty}{\sim} \ln(y)$.

5. a) Let $y \in (-1/\mathrm{e},+\infty)$ and set $X = W(y)+1$. Show that

$$(X-1)\mathrm{e}^X = \mathrm{e}y.$$

b) You are given that

$$\mathrm{e}^X \underset{X\to 0}{=} 1 + X + \frac{X^2}{2} + o(X^2).$$

Deduce that

$$X^2 \underset{y\to(-1/\mathrm{e})^+}{=} 2(\mathrm{e}y+1) + o(X^2).$$

c) Deduce an equivalent of $W+1$ at $(-1/\mathrm{e})^+$.

Exercise 3. We define the function f as

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \arctan(\sinh x). \end{aligned}$$

1. Explain why f is differentiable on $\mathbb{R}$ and determine f' on $\mathbb{R}$.

2. Use the Mean Value Theorem to show that

$$\forall x \in \mathbb{R}_+^*, \quad \frac{x}{\cosh x} < f(x) < x.$$

3. Let $u_0 \in \mathbb{R}_+^*$ and define the sequence $(u_n)_{n\in\mathbb{N}}$ as

$$\forall n \in \mathbb{N}, \quad u_{n+1} = f(u_n).$$

   a) Justify that for all $n \in \mathbb{N}$, $u_n \in \mathbb{R}_+^*$.

   b) Determine the variations of the sequence $(u_n)_{n\in\mathbb{N}}$.

   c) Deduce that the sequence $(u_n)_{n\in\mathbb{N}}$ is convergent.

   d) Determine the limit of the sequence $(u_n)_{n\in\mathbb{N}}$.

4. Show that for all $n \in \mathbb{N}$,

$$\frac{1-\cosh u_n}{u_n \cosh u_n} < \frac{u_{n+1}-u_n}{u_n^2} < 0.$$

5. Show that

$$\lim_{n\to+\infty} \frac{1-\cosh u_n}{u_n \cosh u_n} = 0$$

   and deduce that

$$u_{n+1} \underset{n\to+\infty}{=} u_n + o(u_n^2).$$