INSA LYON

SCAN 1 — Math Test #1 — 1h30

October 18, 2013

No documents, no calculators, no cell phones allowed but you may keep your pet raccoon for moral support.
All your answers must be fully justified, unless noted otherwise.

Exercise 1. 3.5 pts ($\approx$ 2.5 pts)

1. Factor in $\mathbb{R}$ the polynomial function f given by

$$\forall x \in \mathbb{R}, \quad f(x) = 4x^2 - 12x + 8.$$

2. Let $\theta \in \mathbb{R}$. Express (without any justifications) $\sin^2(\theta)$ and $\cos(2\theta)$ in terms of $\cos(\theta)$.
3. Find all the numbers $x \in [-4\pi, 8\pi]$ that satisfy

$$\cos\left(\frac{2x}{3}\right) - 2\sin^2\left(\frac{x}{3}\right) - 12\cos\left(\frac{x}{3}\right) + 11 = 0.$$

Exercise 2. Let $n \in \mathbb{N}$. Compare the numbers 2 pts

$$A = (n!)^2 \qquad \text{and} \qquad B = (2n)!$$

(i.e., say if they are equal or if one is less than the other). Justify your answer.

Exercise 3. Prove by induction that for all $n \in \mathbb{N}$ one has 2.5 pts ✓

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Exercise 4. We define the function f as 4.5 pts ✓

$$\begin{aligned} f\ :\ \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \max\{0, |x| - 1\}. \end{aligned}$$

1. Sketch the graph of f.
2. Show that f is ~~non-increasing~~ on $[-1, +\infty)$. non-decreasing
3. Determine (no justifications required):

$f((0,2])$, $f([-1,1])$, $f(\mathbb{R})$, $f(\{-1,1\})$,
$f^{[-1]}(\mathbb{R}_+^*)$, $f^{[-1]}(\mathbb{R}_-^*)$, $f^{[-1]}(\{0\})$, $f^{[-1]}([1,2])$.

Exercise 5. Let: 5.5 pts

- $n \in \mathbb{N}^*$,
- $\alpha_0, \dots, \alpha_n$ be $n+1$ positive real numbers such that $\alpha_0 + \alpha_1 + \cdots + \alpha_n = 1$,
- $x_0, x_1, \dots, x_n$ be $n+1$ real numbers,

and set:

$$A = \sum_{k=0}^{n} \alpha_k x_k, \qquad B = \sum_{k=0}^{n} \alpha_k x_k^2, \qquad C = \sum_{k=0}^{n} \alpha_k (x_k - A)^2.$$

1. Prove that $C = B - A^2$.

2. From now on we consider the case:

$$\forall k \in \{0, 1, \dots, n\}, \quad x_k = k, \qquad \text{and} \qquad \alpha_k = 2^{-n}\binom{n}{k}.$$

   a) Prove that $\alpha_0 + \alpha_1 + \cdots + \alpha_n = 1$.

   b) You are given that

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1} \qquad \text{and} \qquad \sum_{k=0}^{n} k(k-1)\binom{n}{k} = n(n-1)2^{n-2}.$$

   Deduce the value of A, B, and C.

2 pts ✓

Exercise 6. Let A be a non-empty subset of $\mathbb{R}$ and let $f : A \longrightarrow \mathbb{R}$ be a function. Prove that if f is increasing on A, then f is $1-1$.