SCAN 2 — Solution of Math Test #3

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Exercise 1.

1. For all $n \geq 2$,

$$u_n = \frac{n^2}{n!} = \frac{n}{(n-1)!} = \frac{(n-1)+1}{(n-1)!} = \frac{1}{(n-2)!} + \frac{1}{(n-1)!}.$$

This shows that the series $\sum_n u_n$ converges, as the general term can be written as the sum of the general terms of convergent series. Now,

$$\begin{aligned}\sum_{n=0}^{+\infty} u_n = u_0 + u_1 + \sum_{n=2}^{+\infty} u_n &= 1 + \sum_{n=2}^{+\infty} \frac{1}{(n-2)!} + \frac{1}{(n-1)!}\\ &= 1 + \sum_{n=2}^{+\infty} \frac{1}{(n-2)!} + \sum_{n=2}^{+\infty} \frac{1}{(n-1)!} \qquad \textit{since both series converge}\\ &= 1 + \sum_{n=0}^{+\infty} \frac{1}{n!} + \sum_{n=1}^{+\infty} \frac{1}{n!}\\ &= 2\sum_{n=0}^{+\infty} \frac{1}{n!} = 2\mathrm{e}.\end{aligned}$$

2. Let $n \geq 2$. Then:

$$u_n = \ln\left(1 - \frac{1}{n^2}\right) = \ln\left(\frac{n^2-1}{n^2}\right) = \ln(n-1) + \ln(n+1) - 2\ln n.$$

Now let $N \geq 2$. The corresponding partial sum of the series we're studying is:

$$\begin{aligned}S_N = \sum_{n=2}^{N} u_n &= \sum_{n=2}^{N} \ln(n-1) + \ln(n+1) - 2\ln n\\ &= \sum_{n=2}^{N} \ln(n-1) + \sum_{n=2}^{N} \ln(n+1) - 2\sum_{n=2}^{N} \ln n\\ &= \sum_{n=1}^{N-1} \ln n + \sum_{n=3}^{N+1} \ln n - 2\sum_{n=2}^{N} \ln n\\ &= \ln 1 - \ln N + \sum_{n=2}^{N} \ln n + \ln(N+1) - \ln 2 + \sum_{n=2}^{N} \ln n - 2\sum_{n=2}^{N} \ln n\\ &= \ln\left(\frac{N+1}{N}\right) - \ln 2 \underset{N\to+\infty}{\longrightarrow} -\ln 2.\end{aligned}$$

This shows that the series $\sum_n u_n$ converges and that its sum is $-\ln 2$.

3. We can use the ratio test to show that $\sum_n u_n$ diverges: clearly, all the u_n's are positive, and for all $n \in \mathbb{N}$,

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^{n+1}}{(n+1)!}\,\frac{n!}{n^n} = \left(\frac{n+1}{n}\right)^n \underset{n\to+\infty}{\longrightarrow} \mathrm{e} > 1.$$

Exercise 2. In this exercise we set:

$$\forall n \in \mathbb{N}, \quad \forall x \in \mathbb{R}_+^*, \qquad u_n(x) = \frac{(-1)^n}{1+nx}.$$

1. Let $x \in \mathbb{R}_+^*$. Clearly, the sequence $\big(u_n(x)\big)_{n\in\mathbb{N}}$ is alternating, converges to 0, and has a decreasing absolute value. Hence, by the alternating series test, the series $\sum_n u_n(x)$ converges. Moreover, denoting

$$R_N(x) = \sum_{N+1}^{+\infty} u_n(x),$$

we have:

$$\big|R_N(x)\big| \leq \big|u_{N+1}(x)\big|.$$

Now, let $A > 0$. Since, for a given $n \in \mathbb{N}$, the function $|u_n|$ is non-increasing on $\mathbb{R}_+^*$, we have

$$\|u_n\|_{\infty,[A,+\infty)} = \big|u_n(A)\big| = \frac{1}{1+nA}.$$

Hence, from the inequality obtained just above we have, for all $N \in \mathbb{N}$,

$$\|R_N\|_{\infty,[A,+\infty)} \leq \|u_{N+1}\|_{\infty,[A,+\infty)} \leq \frac{1}{1+(N+1)A} \underset{N\to+\infty}{\longrightarrow} 0.$$

Hence the series of functions $\sum_n u_n$ converges uniformly on $[A,+\infty)$. It is clear that each u_n is continuous on $[A,+\infty)$, hence we conclude that f is continuous on $[A,+\infty)$.

Since for all $A > 0$, f is continuous on $[A,+\infty)$, we conclude that f is continuous on $(0,+\infty)$.

2. From the inequality obtained above, we see that it is sufficient to have

$$\frac{1}{1+(N_a+1)a} < 10^{-3},$$

i.e.,

$$N_a > \frac{999}{a} - 1,$$

so taking $N_a = E(1000/a)$ is sufficient.

Exercise 3.

1. Let $n \in \mathbb{N}^*$. A quick computation shows that for all $x \in \mathbb{R}_+$,

$$f_n'(x) = \frac{x^2\big(3-(2n-3)x^2\big)}{(1+x^2)^{n+1}},$$

hence the function f_n is increasing on $\left[0, \sqrt{3/(2n-3)}\right]$ and decreasing on $\left[\sqrt{3/(2n-3)}, +\infty\right)$. Moreover,

$$f_n(0) = 0, \qquad f_n\left(\sqrt{\frac{3}{2n-3}}\right) = \left(\frac{3}{2n-3}\right)^{3/2} \frac{1}{\big(1+3/(2n-3)\big)^n}, \qquad \lim_{x\to+\infty} f_n(x) = 0.$$

Hence f_n is non-negative, and we have a good idea of the shape of its graph (a bump).

Now, for all $n \in \mathbb{N}^*$,

$$\left(1 + \frac{3}{2n-3}\right)^n = \exp\left(n\ln\left(1+\frac{3}{2n-3}\right)\right)$$

and

$$n\ln\left(1+\frac{3}{2n-3}\right) \underset{n\to+\infty}{\sim} n\frac{3}{2n-3} \underset{n\to+\infty}{\longrightarrow} \frac{3}{2}.$$

Hence,

$$\lim_{n\to+\infty}\left(1+\frac{3}{2n-3}\right)^n = \mathrm{e}^{3/2},$$

so that

$$\lim_{n\to+\infty} f_n\left(\sqrt{\frac{3}{2n-3}}\right) = \lim_{n\to+\infty}\left(\frac{3}{2n-3}\right)^{3/2}\frac{1}{\big(1+3/(2n-3)\big)^n} = 0.$$

We have showed that for all $n \in \mathbb{N}^*$,

$$\|f_n\|_{\infty,\mathbb{R}_+} = \sup_{x\in\mathbb{R}_+}\big|f_n(x)\big| = f_n\left(\sqrt{\frac{3}{2n-3}}\right) \underset{n\to+\infty}{\longrightarrow} 0,$$

hence the sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges uniformly to the nil function on $\mathbb{R}_+$.

Now, let $x \in \mathbb{R}_+^*$. Clearly, the sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $[0,x]$ (as we have uniform convergence on $\mathbb{R}_+$ and $[0,x] \subset \mathbb{R}_+$); moreover, all the functions f_n are continuous on $[0,x]$, hence

$$\lim_{n\to+\infty} F_n(x) = \lim_{n\to+\infty} \int_0^x f_n(t)\,\mathrm{d}t = 0.$$

Clearly, this result is also true for $x = 0$. Hence the sequence of functions $(F_n)_{n\in\mathbb{N}}$ converges pointwise to the nil function on $\mathbb{R}_+$.

2. Since all the functions f_n are non-negative, the functions F_n are non-decreasing on $\mathbb{R}_+$. Hence, for all $n \in \mathbb{N}$,

$$\|F_n\|_{\infty,\mathbb{R}_+} = \lim_{x\to+\infty} F_n(x).$$

Now, for $n \geq 3$ and $x > 0$:

$$\begin{aligned}
F_n(x) &= \int_0^x \frac{t^3}{(1+t^2)^n}\,\mathrm{d}t \\
&= \frac{1}{2}\int_0^{x^2} \frac{s}{(1+s)^n}\,\mathrm{d}t \qquad \textit{using the substitution } s = t^2 \\
&= \frac{1}{2}\int_0^{x^2} \frac{1+s-1}{(1+s)^n}\,\mathrm{d}t \\
&= \frac{1}{2}\int_0^{x^2} \frac{1}{(1+s)^{n-1}}\,\mathrm{d}t - \frac{1}{2}\int_0^{x^2} \frac{1}{(1+s)^n}\,\mathrm{d}t \\
&= \frac{1}{2}\left[-\frac{1}{n-2}\frac{1}{(1+s)^{n-2}}\right]_{s=0}^{s=x^2} + \frac{1}{2}\left[\frac{1}{n-1}\frac{1}{(1+s)^{n-1}}\right]_{s=0}^{s=x^2} \\
&= \frac{1}{2(n-2)}\left(1 - \frac{1}{(1+x^2)^{n-2}}\right) + \frac{1}{2(n-1)}\left(\frac{1}{(1+x^2)^{n-2}} - 1\right) \\
&\underset{x\to+\infty}{\longrightarrow} \frac{1}{2(n-2)} + \frac{1}{2(n-1)}.
\end{aligned}$$

Hence, for all $n \geq 3$,

$$\|F_n\|_{\infty,\mathbb{R}_+} = \frac{1}{2(n-2)} + \frac{1}{2(n-1)} \underset{n\to+\infty}{\longrightarrow} 0.$$

Hence the sequence of functions $(F_n)_{n\in\mathbb{N}}$ converges uniformly to the nil function on $\mathbb{R}_+$.

Exercise 4.

1. Let $n \in \mathbb{N}^*$. A quick study of the function u_n shows that u_n is non-negative on $\mathbb{R}_+$ and increasing on $[0, 1/\sqrt{n}]$ and decreasing on $[1/\sqrt{n}, +\infty)$. Hence,

$$\|u_n\|_{\infty,\mathbb{R}_+} = u_n\left(\frac{1}{\sqrt{n}}\right) = \frac{1}{2n^{3/2}}.$$

By Riemann, the series of functions $\sum_n u_n$ converges normally, hence uniformly on $\mathbb{R}_+$.

2. a) For all $n \in \mathbb{N}^*$, the function u_n is of class C^1 on $\mathbb{R}_+^*$ and

$$\forall x \in \mathbb{R}_+^*, \qquad u_n'(x) = \frac{1-nx^2}{n\big(1+nx^2\big)^2} = \frac{2}{n\big(1+nx^2\big)^2} - \frac{1}{n\big(1+nx^2\big)}.$$

Let $A > 0$. Then

$$\forall x \in [A,+\infty), \qquad \big|u_n'(x)\big| \leq \frac{2}{n\big(1+nx^2\big)^2} + \frac{1}{n\big(1+nx^2\big)} \leq \frac{2}{n^3A^4} + \frac{1}{n^2A^2},$$

hence

$$\|u_n'\|_{\infty,[A,+\infty)} \leq \frac{2}{n^3A^4} + \frac{1}{n^2A^2}$$

which is the general term of a convergent series (sum of two convergent Riemann series). Hence $\sum_n u_n'$ converges normally, hence uniformly on $[A,+\infty)$. Moreover, $\sum_n u_n$ converges at any $x_0 \in [A,+\infty)$, hence, by the term by term differentiation theorem, f is of class C^1 on $[A,+\infty)$ and

$$\forall x \in [A,+\infty), \qquad f'(x) = \sum_{n=1}^{+\infty} \frac{1-nx^2}{n\big(1+nx^2\big)^2}.$$

This is true for all $A \in (0,+\infty)$, hence f is of class C^1 on $\mathbb{R}_+^*$ and

$$\forall x \in \mathbb{R}_+^*, \qquad f'(x) = \sum_{n=1}^{+\infty} \frac{1-nx^2}{n\big(1+nx^2\big)^2}.$$

b) Let $x \in \mathbb{R}_+^*$ and define the function h on $\mathbb{R}_+^*$ by $h(t) = x/\big(t(1+tx^2)\big)$. Clearly, the function h is positive, continuous and decreasing on $\mathbb{R}_+^*$, and for all $n \in \mathbb{N}^*$, $u_n(x) = h(n)$, hence, by the integral comparison test,

$$\int_1^{+\infty} h(t)\,\mathrm{d}t \leq f(x) = \sum_{n=1}^{+\infty} h(n) = \frac{x}{1+x^2} + \sum_{n=2}^{+\infty} h(n) \leq \frac{x}{1+x^2} + \int_1^{+\infty} h(t)\,\mathrm{d}t,$$

Now,

$$\begin{aligned}
\int_1^{+\infty} h(t)\,\mathrm{d}t &= \int_1^{+\infty} \frac{x}{t\big(1+tx^2\big)}\,\mathrm{d}t \\
&= \int_1^{+\infty} \left(\frac{x}{t} - \frac{x^3}{1+tx^2}\right)\mathrm{d}t \\
&= \left[x\ln\left(\frac{t}{1+tx^2}\right)\right]_{t=1}^{t\to+\infty} \\
&= x\ln\left(\frac{1}{x^2}\right) - x\ln\left(\frac{1}{1+x^2}\right) \\
&= -x\ln(x^2) + x\ln(1+x^2) \\
&= x\ln\left(\frac{1+x^2}{x^2}\right),
\end{aligned}$$

hence the result. From this we conclude:

i) For all $x \in \mathbb{R}_+^*$,

$$\ln\left(\frac{1+x^2}{x^2}\right) \leq \frac{f(x)-f(0)}{x-0} = \frac{f(x)}{x} \leq \frac{1}{1+x^2} + \ln\left(\frac{1+x^2}{x^2}\right)$$

hence

$$\lim_{x\to0^+} \frac{f(x)-f(0)}{x-0} = +\infty$$

hence f is not differentiable (from the right) at 0.

ii) For all $x \in (0,1)$,

$$0 \leq x\ln\left(\frac{1+x^2}{x^2}\right) \leq x\ln\left(\frac{2}{x^2}\right) = x\ln 2 - 2x\ln x \underset{x\to0^+}{\longrightarrow} 0.$$

Hence, by the Squeeze Theorem, $\lim_{x\to0^+} f(x) = 0$.

iii) Clearly,

$$x\ln\left(\frac{1+x^2}{x^2}\right) = x\ln\left(1+\frac{1}{x^2}\right) \underset{x\to+\infty}{\sim} x\frac{1}{x^2} = \frac{1}{x} \underset{x\to+\infty}{\longrightarrow} 0.$$

By the Squeeze Theorem, $\lim_{x\to+\infty} f(x) = 0.$