SCAN 2 — Solution of Math Test #2

Romaric Pujol, romaric.pujol@insa-lyon.fr

December 12, 2013

Exercise 1. Clearly, the function f is of class C^2 on $\mathbb{R}^2$, hence f admits a second order Taylor–Young expansion at $(1,2)$. For all $(x,y)\in\mathbb{R}^2$,

$$\begin{aligned}
\partial_1 f(x,y) &= (x+1)\mathrm{e}^{x+2y}, & \partial_1 f(1,2) &= 2\mathrm{e}^5,\\
\partial_2 f(x,y) &= 2x\mathrm{e}^{x+2y}, & \partial_2 f(1,2) &= 2\mathrm{e}^5,\\
\partial^2_{1,1} f(x,y) &= (x+2)\mathrm{e}^{x+2y}, & \partial^2_{1,1} f(1,2) &= 3\mathrm{e}^5,\\
\partial^2_{1,2} f(x,y) &= 2(x+1)\mathrm{e}^{x+2y}, & \partial^2_{1,2} f(1,2) &= 4\mathrm{e}^5,\\
\partial^2_{2,2} f(x,y) &= 4x\mathrm{e}^{x+2y}, & \partial^2_{2,2} f(1,2) &= 4\mathrm{e}^5.
\end{aligned}$$

Hence the second order Taylor–Young formula of f at $(1,2)$ is:

$$f(x,y) \underset{(x,y)\to(1,2)}{=} \mathrm{e}^5 + 2\mathrm{e}^5(x-1) + 2\mathrm{e}^5(y-2) + \frac{3}{2}\mathrm{e}^5(x-1)^2 + 4\mathrm{e}^5(x-1)(y-2) + 2\mathrm{e}^5(y-2)^2 + o\big(\|(x-1,y-2)\|^2\big).$$

Exercise 2.

1. The function g is of class C^1 on $\mathbb{R}^2$ as the composition of the linear mapping $\varphi : (u,v)\mapsto(2u+v,u+v)$ and of the function f which is of class C^1 on $\mathbb{R}^2$. Moreover, for all $(u,v)\in\mathbb{R}^2$, by the chain rule:

$$\begin{aligned}
\partial_1 g(u,v) &= 2\partial_1 f(2u+v,u+v) + \partial_2 f(2u+v,u+v),\\
\partial_2 g(u,v) &= \partial_1 f(2u+v,u+v) + \partial_2 f(2u+v,u+v).
\end{aligned}$$

2. The linear mapping $\varphi : (u,v)\mapsto(2u+v,u+v)$ is a bijection from $\mathbb{R}^2$ to $\mathbb{R}^2$ and its inverse is: $\varphi^{-1} : (x,y)\mapsto (x-y,-x+2y)$. Let $(x,y)\in\mathbb{R}^2$. Then:

$$\partial_1 f(x,y) + \partial_2 f(x,y) + (2y-x)f(x,y) = \partial_2 g(x-y,-x+2y) + (2y-x)g(x-y,-x+2y).$$

Hence f is a solution of Equation (E) on $\mathbb{R}^2$ if and only if:

$$\forall (x,y)\in\mathbb{R}^2,\quad \partial_2 g(x-y,-x+2y) + (2y-x)g(x-y,-x+2y) = 0.$$

Since φ is a bijection from $\mathbb{R}^2$ to $\mathbb{R}^2$, this is also equivalent to:

$$\forall (u,v)\in\mathbb{R}^2,\quad \partial_2 g(u,v) + vg(u,v) = 0.$$

3. We can consider the auxiliary differential equation:

$$y'(u) + vy(u) = 0$$

(where the variable is u and v is a constant). The general solution of this differential equation is

$$y(u) = A\mathrm{e}^{-vu}.$$

Hence the general solution of class C^1 of Equation (E′) on $\mathbb{R}^2$ is:

$$g(u,v) = A(v)\mathrm{e}^{-vu}$$

where A is a function of class C^1 on $\mathbb{R}$.

4. Hence, f is a solution of class C^1 on $\mathbb{R}^2$ of Equation (E) if and only if f has the form

$$f(x,y) = A(-x+2y)\mathrm{e}^{(x-2y)(x-y)}$$

where A is of class C^1 on $\mathbb{R}$.

Exercise 3.

1. a) Since U is an open set in $(E,\|\cdot\|)$ and since $a_0\in U$, there exists $r>0$ such that the open ball centered at a_0 of radius r (denoted by $\mathring{B}_r(a_0)$) is included in U. If $h=0_E$, take $\varepsilon=42>0$ and if $h\neq 0_E$, take $\varepsilon = r/\|h\|>0$. With this ε we do have $\forall t\in(-\varepsilon,\varepsilon)$, $a_0+th\in\mathring{B}_r(a_0)$ and hence $a_0+th\in U$.

   b) Let $t\in(0,\varepsilon)$. Then, since $f_1\leq f_2$ throughout U and since $f_1(a_0)=f_2(a_0)$ we have:

$$f_1(a_0+th) - f_1(a_0) \leq f_2(a_0+th)f_2(a_0)$$

   and since $t>0$,

$$\frac{f_1(a_0+th)-f_1(a_0)}{t} \leq \frac{f_2(a_0+th)f_2(a_0)}{t}.$$

   Now taking the limit as $t\to 0^+$ (which is valid since $(0,\varepsilon)$ is a right-sided punctured neighborhood of 0) yields, by definition of the directional derivative,

$$\nabla_h f_1(a_0) \leq \nabla_h f_2(a_0).$$

   c) If $t\in(-\varepsilon,0)$, we'll obtain:

$$\frac{f_1(a_0+th)-f_1(a_0)}{t} \geq \frac{f_2(a_0+th)f_2(a_0)}{t},$$

   and taking the limit as $t\to 0^-$ yields:

$$\nabla_h f_1(a_0) \geq \nabla_h f_2(a_0).$$

2. We proved that for all $h\in E$, $\nabla_h f_1(a_0) = \nabla_h f_2(a_0)$ or, equivalently, for all $h\in E$, $\mathrm{d}_{a_0} f_1(h) = \mathrm{d}_{a_0} f_2(h)$. Hence $\mathrm{d}_{a_0} f_1 = \mathrm{d}_{a_0} f_2$.

3. First observe that $g(a_0) = f_1(a_0) = f_2(a_0)$. Hence, from the assumptions and the fact that $\mathrm{d}_{a_0} f_1 = \mathrm{d}_{a_0} f_2$, we obtain,

$$\forall x\in U,\quad f_1(x) - f_1(a_0) - \mathrm{d}_{a_0} f_1(x-a_0) \leq g_1(x) - g_1(a_0) - \mathrm{d}_{a_0} f_1(x-a_0) \leq f_2(x) - f_2(a_0) - \mathrm{d}_{a_0} f_2(x-a_0)$$

   and hence for all $x\in U\setminus\{a_0\}$,

$$\frac{f_1(x)-f_1(a_0)-\mathrm{d}_{a_0} f_1(x-a_0)}{\|x-a_0\|} \leq \frac{g_1(x)-g_1(a_0)-\mathrm{d}_{a_0} f_1(x-a_0)}{\|x-a_0\|} \leq \frac{f_2(x)-f_2(a_0)-\mathrm{d}_{a_0} f_2(x-a_0)}{\|x-a_0\|}.$$

   Now, since:

$$\lim_{\|x-a_0\|\to 0} \frac{f_1(x)-f_1(a_0)-\mathrm{d}_{a_0} f_1(x-a_0)}{\|x-a_0\|} = 0$$

   and

$$\lim_{\|x-a_0\|\to 0} \frac{f_2(x)-f_2(a_0)-\mathrm{d}_{a_0} f_2(x-a_0)}{\|x-a_0\|} = 0$$

   we obtain, by the Squeeze Theorem:

$$\lim_{\|x-a_0\|\to 0} \frac{g(x)-g(a_0)-\mathrm{d}_{a_0} f_1(x-a_0)}{\|x-a_0\|} = 0.$$

   The mapping $\mathrm{d}_{a_0} f_1$ being linear and continuous on E, we can conclude, from the definition of differentiability, that g is differentiable at a_0 and that $\mathrm{d}_{a_0} g = \mathrm{d}_{a_0} f_1$.

Exercise 4. First observe that $M_2(\mathbb{R})$ is a finite-dimensional vector space over $\mathbb{R}$ (as it is of dimension 4), hence all the norms on $M_2(\mathbb{R})$ are equivalent. We'll use the norm n defined by:

$$n\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right) = |a| + |b| + |c| + |d|.$$

This mapping n is indeed a norm on $M_2(\mathbb{R})$ as the mapping

$$\begin{aligned}
\varphi :\ M_2(\mathbb{R}) &\longrightarrow \mathbb{R}^4\\
\begin{pmatrix} a & b\\ c & d\end{pmatrix} &\longmapsto (a,b,c,d)
\end{aligned}$$

is a *linear bijection* and clearly, for all A, $n(A) = \|\varphi(A)\|_1$.

Let $h \in M_2(\mathbb{R})$, say $h = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Then:

$$\begin{aligned} \det(I+h) &= \det \begin{pmatrix} 1+\alpha & \beta \\ \gamma & 1+\delta \end{pmatrix} \\ &= (1+\alpha)(1+\delta) - \beta\gamma \\ &= 1 + \alpha + \delta + (\alpha\delta - \beta\gamma) \\ &= 1 + \mathrm{tr}(h) + (\alpha\delta - \beta\gamma). \end{aligned}$$

Hence,

$$\left|\det(I+h) - \det(I) - \mathrm{tr}(h)\right| = |\alpha\delta - \beta\gamma| \leq |\alpha\delta| + |\beta\gamma| \leq 2n(h)^2.$$

Hence:

$$\frac{\left|\det(I+h) - \det(I) - \mathrm{tr}(h)\right|}{n(h)} \leq 2n(h) \underset{n(h)\to 0}{\longrightarrow} 0.$$

Now, tr is clearly linear (and continuous since we're in a finite-dimensional vector space). Hence det is differentiable at I and $\mathrm{d}_I \det = \mathrm{tr}$.