SCAN 2 — Solution of Math Test #1

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Exercise 1. Let $A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$. The eigenvalues of A are:

- -3 of multiplicity 2,
- 0 of multiplicity 1.

The associated eigenvectors are

$$X_{-3} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \qquad X'_{-3} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \qquad X_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Hence, if we set:

$$P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we have $A = PDP^{-1}$ and hence:

$$X \text{ is a solution of (S)} \iff X'(t) = AX(t) \iff Y'(t) = DY(t)$$

where $Y(t) = P^{-1}X(t)$. Denote by $u(t)$, $v(t)$ and $w(t)$ the components of $Y(t)$, i.e.,

$$Y(t) = \begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix}$$

then:

$$X \text{ is a solution of (S)} \iff \begin{cases} u'(t) = -3u(t) \\ v'(t) = -3v(t) \\ w'(t) = \quad 0 \end{cases} \iff \exists (A,B,C) \in \mathbb{R}^3, \quad \begin{cases} u(t) = Ae^{-3t} \\ v(t) = Be^{-3t} \\ w(t) = \quad C \end{cases}$$

Hence, the general solution of (S) is:

$$\begin{cases} x(t) = -Ae^{-3t} - Be^{-3t} + C \\ y(t) = \quad Ae^{-3t} \qquad\quad + C \\ z(t) = \qquad\qquad Be^{-3t} + C, \end{cases} \qquad \text{with } A, B, C \in \mathbb{R}.$$

Exercise 2. Equation $(*)$ is a first order linear differential equation with non-constant coefficients a and b where $a(x) = x$ and $b(x) = x^2 + 1$. Since a doesn't vanish on $(0, +\infty)$, we can solve Equation $(*)$ on that interval. We now need an antiderivative A of $-b/a$ on $(0, +\infty)$:

$$A(x) = \int -\frac{x^2+1}{x}\,\mathrm{d}x = -\int x + \frac{1}{x}\,\mathrm{d}x = -\frac{x^2}{2} - \ln(x) + \text{constant}$$

and we'll take $A(x) = -\dfrac{x^2}{2} - \ln(x)$. The general solution of the associated homogeneous equation is

$$y_H(x) = Ke^{A(x)} = K\frac{e^{-x^2/2}}{x}, \quad K \in \mathbb{R}.$$

It should be clear that the constant function 1 is a particular solution of $(*)$ on $(0, +\infty)$, hence the general solution of $(*)$ on $(0, +\infty)$ is:

$$y(x) = K\frac{e^{-x^2/2}}{x} + 1, \quad K \in \mathbb{R}.$$

The solution of $(*)$ satisfying $y(1) = \sqrt{e} + 1$ is obtained by taking K such that:

$$Ke^{-1/2} + 1 = \sqrt{e} + 1,$$

i.e., $K = e$. Hence, the solution y of $(*)$ on $(0, +\infty)$ such that $y(1) = \sqrt{e} + 1$ is:

$$y(x) = e^{1-x^2/2} + 1.$$

Exercise 3.

1. N is a norm on $\mathbb{R}^2$:
   - Observe that N only takes non-negative values.
   - Let $(x, y) \in \mathbb{R}^2$. Then:
   $$\begin{aligned} N(x,y) = 0 &\iff |2x+y| + |x+y| = 0 \\ &\iff \begin{cases} 2x + y = 0 \\ x + y = 0 \end{cases} \\ &\iff (x,y) = (0,0). \end{aligned}$$
   - Let $\lambda \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2$. Then:
   $$N\big(\lambda(x,y)\big) = N(\lambda x, \lambda y) = |2\lambda x + \lambda y| + |\lambda x + \lambda y| = |\lambda| N(x,y).$$
   - Let $(x,y), (x',y') \in \mathbb{R}$. Then:
   $$\begin{aligned} N\big((x,y) + (x',y')\big) &= N(x + x', y + y') \\ &= \big|2(x+x') + (y+y')\big| + \big|(x+x') + (y+y')\big| \\ &= |2x + y \ + \ 2x' + y'| + |x + y \ + \ x' + y'| \\ &\leq |2x+y| + |2x'+y'| + |x+y| + |x'+y'| \\ &= N(x,y) + N(x',y'). \end{aligned}$$

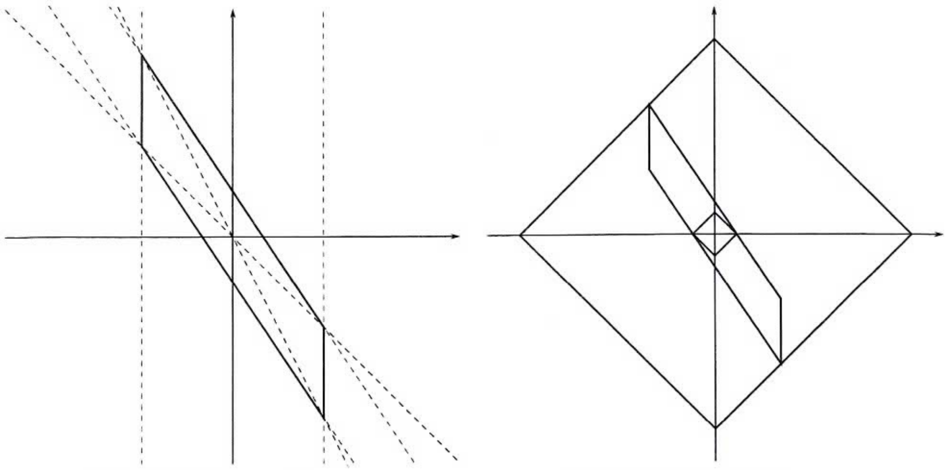
   Hence N is a norm on $\mathbb{R}^2$.

2. Let $(x, y) \in \mathbb{R}^2$. We have four cases:
   - If $2x + y \geq 0$ and $x + y \geq 0$, i.e., $y \geq -2x$ and $y \geq -x$:
   $$(x,y) \in \overline{B} \iff 2x + y + x + y \leq 1 \iff 3x + 2y \leq 1 \iff y \leq -\frac{3}{2}x + \frac{1}{2}.$$
   - If $2x + y \geq 0$ and $x + y \leq 0$, i.e., $y \geq -2x$ and $y \leq -x$:
   $$(x,y) \in \overline{B} \iff 2x + y - x - y \leq 1 \iff x \leq 1.$$

   We now have have one half of $\overline{B}$, the other half can be deduced from the symmetry with respect to $(0,0)$. See Figure 1a for the picture of the ball. The ball $\overline{B}$ is the parallelogram with vertices $(-1,2)$, $(1,-1)$, $(1,-2)$ and $(-1,1)$.

3. Since $\mathbb{R}^2$ is a finite-dimensional vector space, all the norms on $\mathbb{R}^2$ are equivalent, hence N and $\|\cdot\|_1$ are equivalent.

4. We know that the balls for $\|\cdot\|_1$ are squares with diagonals parallel to the axes.
   - Now, $\big\|(1,-2)\big\|_1 = 3$ and $\big\|(1,-1)\big\|_1 = 2$, so if we take the ball $\overline{B}_3$ for $\|\cdot\|_1$ of radius 3, we'll have $\overline{B} \subset \overline{B}_3$. This means that $\alpha = 1/3$ works. (We're implicitly using the convexity of the balls). Or, formally,
   $$\begin{aligned} \frac{1}{3}\big\|(x,y)\big\|_1 &= \frac{1}{3}\big(|x| + |y|\big) \\ &= \frac{1}{3}\big(|2x + y - x - y| + |2x + 2y - 2x - y|\big) \\ &\leq \frac{1}{3}\big(|2x+y| + |x+y| + 2|x+y| + |2x+y|\big) \\ &= \frac{1}{3}\big(2|2x+y| + 3|x+y|\big) \\ &\leq |2x+y| + |x+y| = N(x,y). \end{aligned}$$



(a) Unit ball for the norm N (b) Inclusions: $\overline{B_{1/3}} \subset \overline{B} \subset \overline{B_3}$

Figure 1. Figures for Exercise 3

- Now, we notice that the intersection of the unit sphere of N with the axes are the points $(\pm 1/3, 0)$ and $(0, \pm 1/2)$. Hence, the closed ball $\overline{B_{1/3}}$ for $\|\cdot\|_1$ of radius $1/3$ is included in $\overline{B}$, and hence $\beta = 3$ works. (We're implicitly using the convexity of the balls). Or, formally,

$$\begin{aligned} N(x,y) &= |2x+y| + |x+y| \\ &\leq 2|x| + |y| + |x| + |y| \\ &= 3|x| + 2|y| \\ &\leq 3\big(|x| + |y|\big) = 3\big\|(x,y)\big\|_1. \end{aligned}$$

See Figure 1b for these inclusions.

Exercise 4.

1. For all $x \in [0,1]$, $f'(x) = ne^{-nx}(1-nx)$, hence f is increasing on $[0, 1/n]$ and decreasing on $[1/n, 1]$. The maximal value of f is attained at $1/n$, and $f(1/n) = 1/\mathrm{e}$. See Figure 2.

2. Let $x \in (0,1]$. Then: $\lim\limits_{n\to+\infty} f_n(x) = 0$ since we know that $\lim\limits_{X\to+\infty} X\mathrm{e}^{-X} = 0$. Moreover, $\lim\limits_{n\to+\infty} f_n(0) = 0$ since for all $n \in \mathbb{N}^*$, $f_n(0) = 0$. Hence:

$$\forall x \in [0,1], \quad f(x) = \lim_{n\to+\infty} f_n(x) = 0.$$

The function f is clearly a real-valued continuous function on $[0,1]$, hence $f \in E$.

3. Let $n \in \mathbb{N}^*$:

$$\|f_n\|_\infty = \frac{1}{\mathrm{e}} \qquad \textit{(as determined in Question 1)}$$

$$\|f_n\|_1 = \int_0^1 nx\mathrm{e}^{-nx}\,\mathrm{d}x = \frac{1 - \mathrm{e}^{-n} - n\mathrm{e}^{-n}}{n}.$$

4. a) We see that the maximal value of $|f_n|$ on $[0,1]$ doesn't approach 0 as $n \to +\infty$ (the bumps keep the same height), so $(f_n)_{n\in\mathbb{N}^*}$ doesn't approach the nil function for the norm $\|\cdot\|_\infty$.

   b) As shown in Question 1, $\|f_n - f\|_\infty = \|f_n\|_\infty = 1/\mathrm{e}$ which doesn't approach 0 as $n \to +\infty$, hence $(f_n)_{n\in\mathbb{N}^*}$ doesn't converge to f for the norm $\|\cdot\|_\infty$.

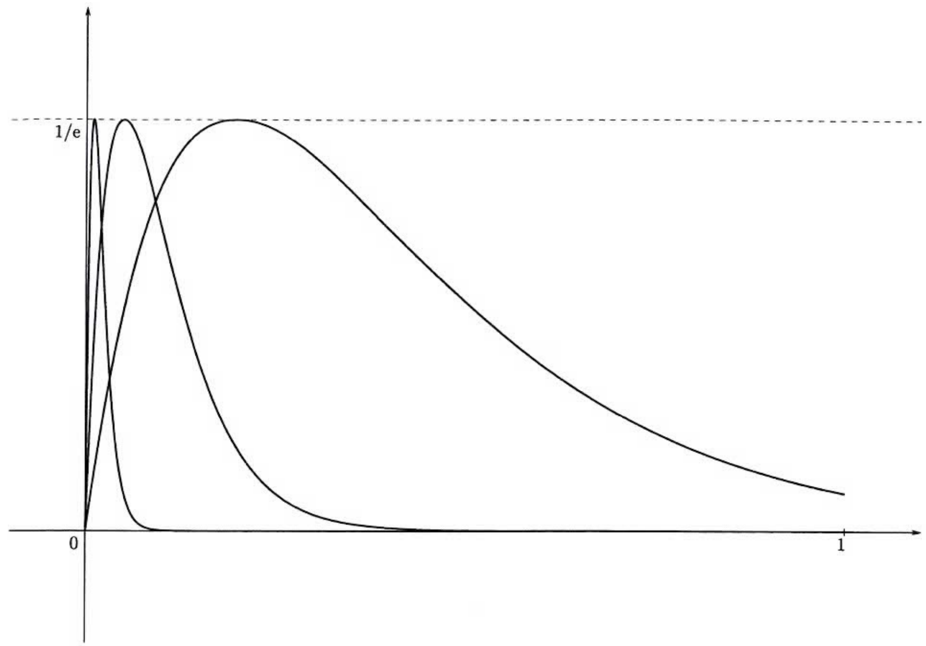


Figure 2. Graph of f_5, f_{20} and f_{100} of Exercise 4

5. a) We observe that the bumps formed by the graphs of the f_n's get narrower as $n \to +\infty$, while remaining at a constant height. It really seems that the surface area enclosed by these bumps approaches 0 as $n \to +\infty$.

   b) As shown in Question 3,

$$\|f_n - f\|_1 = \|f_n\|_1 = \frac{1 - \mathrm{e}^{-n} - n\mathrm{e}^{-n}}{n} \underset{n\to+\infty}{\longrightarrow} 0.$$

   Hence the limit of the sequence $(f_n)_{n\in\mathbb{N}^*}$ is f for the norm $\|\cdot\|_1$.

6. The norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are of course *not* equivalent, as for the norm $\|\cdot\|_1$, $(f_n)_{n\in\mathbb{N}^*}$ converges to f, but not form the norm $\|\cdot\|_\infty$.