Interrogations Écrites
SCAN - Deuxième année

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SCAN 2 — Test #1 — 1h30 *November 12, 2009*

No document, no calculator, no mobile-phone allowed.

Exercise 1. All questions are independant.

1. Show that a continuous linear map is differentiable. What is its differential?
2. Show that the function
$$f(x,y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \dfrac{x^2y^2}{x^2+y^2} & \text{otherwise.} \end{cases}$$
is continuous on $\mathbb{R}^2$.
3. Recall the definition of equivalent norms.
4. Show that $N(x,y) = |x| + |2y|$ defines a norm on $\mathbb{R}^2$ and sketch the unit ball.

Exercise 2. Let $\omega \in \mathbb{R}_+^*$. Solve
$$y'' + 4y = \sin \omega t.$$

Exercise 3. Give the real form of solutions of the linear differential system
$$\begin{cases} x' = -x + 2y \\ y' = z \\ z' = 2x - 3y + 2z. \end{cases}$$
Deduce the real solutions of
$$\begin{cases} x' = -x + 2y \\ y'' - 2y' = 2x - 3y \\ x(0) = 1,\ y(0) = y'(0) = 0. \end{cases}$$

Exercise 4. In this exercise we study differential equations of the form

(E) $$y'(x) = g\left(\frac{y}{x}\right)$$

on an interval I with $0 \notin I$, where g is a continuous function.

1. Set $u(x) = \dfrac{y(x)}{x}$. Show that (E) is equivalent to:

   (E') $$u'(x) = \big(g\big(u(x)\big) - u(x)\big)\frac{1}{x}.$$

   How are the equilibrium solutions of (E') determined?
2. Consider the following problem:

   (P) $$\begin{cases} y'(x) = \dfrac{x+2y}{2x+y} \\ y(1) = 0. \end{cases}$$

   Explain why problem (P) has a unique solution in an open interval containing 1.
3. Show that (P) can be written as:
$$\begin{cases} y'(x) = g\left(\dfrac{y}{x}\right) \\ y(1) = 0. \end{cases}$$
   for a function g you will explicit.
4. Give the equation of the solution curve of (P) in this case (don't try to explicit the solution).

SCAN 2 — Test #2 — 1h30 *December 17, 2009*

No document, no calculator, no mobile-phone allowed.

Exercise 1. In this exercise, all questions are independant.

1. What is a geometric series? When does it converge? When does it diverge? When it converges, what is its sum?
2. Let $(u_n)_{n\geq 0}$ be a sequence of real numbers. For all $n \in \mathbb{N}$ we set $v_n = u_{n+1} - u_n$. Under what condition is the series
$$\sum_{n=0}^{\infty} v_n$$
convergent? Compute its value in this case.
3. Let g a function of class C^2 on $\mathbb{R}$ and let $f(x,y) = g\left(\frac{y^2}{x}\right)$. Where is the function f of class C^2?
Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$.
4. Where do the following relations define a local diffeomorphism?
$$\begin{cases} x = u^3 + v^3 \\ y = uv. \end{cases}$$
Compute $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$.

Exercise 2. We consider the following partial differential equations:

$$(\mathrm{E}_0) \qquad x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = 0,$$

$$(\mathrm{E}_1) \qquad x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} + x^2 f(x,y) = 0.$$

1. Show that the following change of variables can be used to solve (E_0) and (E_1) on $U = \mathbb{R}_+^* \times \mathbb{R}_+^*$:
$$\begin{cases} s = \frac{x}{y} \\ t = y. \end{cases}$$
2. Find all the functions of class C^2 on U that solve Equation (E_0).
3. Find all the functions of class C^2 on U that solve Equation (E_1) as well as the following initial conditions:
$$f(\pi, y) = 0 \text{ and } f\left(\frac{\pi}{2}, y\right) = y, \ \forall y \in \mathbb{R}_+^*.$$

Exercise 3. Let P and Q be polynomials with real coefficients and ε a real number. If a is a root of P and if ε is small enough, it is legitimate to conjecture that $P + \varepsilon Q$ will have a root $\varphi(\varepsilon)$ close to a. The goal of this exercise is to study the roots of the perturbed polynomial $P + \varepsilon Q$ on a particular example.
We set $P = X^2 - 4$, $Q = X^3$, and $a \in \{-2, 2\}$ is a root of P. We also set

$$f(x, \varepsilon) = (x^2 - 4) + \varepsilon x^3.$$

1. Apply the implicit function theorem to the function f at the point $(a, 0)$ and show that the equation

   $$(x^2 - 4) + \varepsilon x^3 = 0 \tag{1}$$

   defines x in terms of ε in a neighborhood of $(a, 0)$. We denote by $x_a(\varepsilon)$ this solution.
2. Of what differentiability class is the function x_a?
3. Give the Taylor expansion of order 2 of $x_a(\varepsilon)$ in a neighborhood of 0 for $a = -2$ and $a = 2$.
4. Say why, for ε sufficiently close to 0 but not 0, the three solutions of Equation (1) are real. We denote by $x_3(\varepsilon)$ the third root.
5. Show that
   $$x_{-2}(\varepsilon) + x_2(\varepsilon) + x_3(\varepsilon) = -\frac{1}{\varepsilon}.$$
6. Deduce a generalized expansion of order 2 at 0 of $x_3(\varepsilon)$.
7. Explain why for $\varepsilon > 0$ small enough, the three roots of Equation (1) are in the following order:

   $$x_3(\varepsilon) < x_{-2}(\varepsilon) < x_2(\varepsilon).$$

   In what order are the roots of Equation (1) if $\varepsilon < 0$ and close enough to 0?

SCAN 2 — Test #3 — 1h30 *March 25, 2010*

No document, no calculator, no mobile-phone allowed.

Exercise 1. We consider the following series of functions:

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n + n^2 x^2}.$$

1. Give the domain (in $\mathbb{R}$) of the function f.
2. Show that for all $a > 0$ the series f converges uniformly on $[a, +\infty)$. Deduce that f is continuous on $(0, +\infty)$.
3. Without trying to differentiate it, show that the function f is decreasing on $(0, \infty)$.
4. Show that f can be differentiated term by term on $(0, +\infty)$.
5. Find a constant A such that
$$\forall x > 0,\ f(x) \leq \frac{A}{x^2}.$$
Deduce the value of the limit $\lim_{x \to +\infty} f(x)$.
6. Prove that for all $x > 0$,
$$f(x) \geq \int_1^{+\infty} \frac{dt}{t(1 + tx^2)} = \ln\left(1 + \frac{1}{x^2}\right).$$
Deduce the value of the limit $\lim_{x \to 0^+} f(x)$.

Exercise 2. We recall that the notation $C\big([0,1]\big)$ stands for the set of real-valued continuous functions on $[0, 1]$.
Let $\alpha \in \mathbb{R}$. The goal of this exercise is to study the solutions of the following differential problem:

$$\text{(D)} \qquad \begin{cases} \forall x \in [0,1],\ f'(x) = \sin\big(xf(x)\big) \\ f(0) = \alpha. \end{cases}$$

1. Preliminary question: Show that for all real numbers a and b, the following inequality holds true:
$$|\sin b - \sin a| \leq |b - a|.$$
2. Find an integral equation in $C\big([0,1]\big)$ that is equivalent to System (D).
3. Use the Fixed Point Theorem to show that the integral equation from the previous question has a unique solution in $C\big([0,1]\big)$.
4. Explicit a sequence of functions (f_n) that approaches uniformly the solution f of System (D). Give an estimate of $\|f - f_n\|_\infty$. Compute $f_2(x)$ when $\alpha = 1$ and $\forall x \in [0,1],\ f_0(x) = 0$.

Exercise 3. In this exercise, the two parts are independant.

Part I

Let $A \in M_n(\mathbb{C})$ and B an element of $\mathbb{C}^n$.

1. Use the Fixed Point Theorem to show that if there is a subordinate matrix norm such that $|||A||| < 1$ then the linear system

   $$X = AX + B \tag{1}$$

   of unknown X has a unique solution in $\mathbb{C}^n$, and explicit a sequence of elements of $\mathbb{C}^n$ that converges to the fixed point.
2. Show that $\rho(A) \leq |||A|||$, where $\rho(A)$ is the spectral radius of A defined as:

   $$\rho(A) = \max\left\{|\lambda|;\ \lambda \in \mathbb{C} \text{ is an eignenvalue of } A\right\}.$$

3. In this question only we study two special cases:

   a) In this question only we consider

   $$A = \frac{1}{8}\begin{pmatrix} -4 & 1 & 1 \\ -1 & 1 & 2 \\ 2 & -2 & 1 \end{pmatrix}.$$

   Show that System (1) as a unique fixed point and that the sequence you defined in Question 1 converges.

   b) In this question only we consider

   $$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

   Show that we cannot apply the method from Question 1 in this case.

Part II

In this part we assume that $A \in M_n(\mathbb{C})$ is a diagonalizable matrix, that is, there exists an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

1. Show that if $\|\cdot\|$ is a norm on $\mathbb{C}^n$ then $\|\cdot\|_P$ defined as

   $$\forall X \in \mathbb{C}^n,\ \|X\|_P = \left\|P^{-1}X\right\|$$

   is also a norm on $\mathbb{C}^n$.
2. We denote by $|||\cdot|||_P$ the matrix norm subordinate to $\|\cdot\|_P$. Show that $|||A|||_P = |||D|||$.
3. Deduce that there exists a subordinate norm N such that $N(A) = \rho(A)$.

SCAN 2 — Test #4 — 1h30 *May 7, 2010*

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Exercise 1.

1. Give the definition of a scalar product.
2. In the vector space $\mathbb{R}^2$ we define the following quadratic form:

$$q(x, y) = x^2 + y^2 + xy.$$

   a) Give the matrix associated with the quadratic form q in the canonical basis of $\mathbb{R}^2$.

   b) We consider the following basis of $\mathbb{R}^2$:

$$\mathcal{B} = \big((1, 1), (1, -1)\big).$$

   Give the matrix associated with the quadratic form q in the basis $\mathcal{B}$.

   c) Compute $\varphi\big((1, 1), (1, -1)\big)$, where φ is the polar form associated with the quadratic form q.

Exercise 2.

1. Give the radius of convergence of the power series

$$\sum_{n=1}^{+\infty} n x^n$$

   and compute its sum.

2. We consider the following rational function:

$$\varphi(x) = \frac{x}{(1-x)^2\big(1+x^2\big)}.$$

   Give the power series expansion of φ, and specify the domain of validity.

3. We want to solve the following system of sequences, for $n \in \mathbb{N}$:

$$\begin{cases} a_{n+1} + b_n = n \\ a_n - b_{n+1} = 1 \\ a_0 \in \mathbb{R},\ b_0 = -1. \end{cases}$$

   We consider the following power series:

$$f(x) = \sum_{n=0}^{+\infty} a_n x^n.$$

   a) Show that

$$f(x) = \frac{a_0 + x(1 - 2a_0) + a_0 x^2}{\big(1+x^2\big)(1-x)^2}.$$

   b) In the case $a_0 = 0$, deduce an expression of a_n and b_n in terms of n.

Exercise 3. Let f be the function defined by:

$$f(x) = 1 + \pi - \frac{8}{\pi} \sum_{n=0}^{+\infty} \frac{\cos(2n+1)x}{(2n+1)^2}.$$

1. a) Show that f is continuous on $\mathbb{R}$.
   b) Show, using Fourier series, that for all $x \in [0, \pi]$,

   $$f(x) = 2x + 1.$$

   c) Sketch the graph of the function f on the interval $[-2\pi, 2\pi]$.
   d) Give the value of the sum of the series:

   $$\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2}.$$

2. Let $n \in \mathbb{N}$ and $\alpha_n \in \mathbb{R}$. We consider the following linear differential equation

   $$(E_n) \qquad y'' + 4y' + y = \alpha_n \cos nt.$$

   a) Show that the function y_n defined by

   $$\forall t \in \mathbb{R}, \; y_n(t) = A_n \cos nt + B_n \sin nt$$

   is a solution of Equation (E_n) if

   $$A_n = \frac{\alpha_n \left(1 - n^2\right)}{\left(1 - n^2\right)^2 + 16n^2}, \qquad B_n = \frac{4n\alpha_n}{\left(1 - n^2\right)^2 + 16n^2}.$$

   b) Give a simple equivalent of A_n and B_n when $n \to +\infty$.
   c) Let $(\alpha_n)_{n \geq 0}$ be a sequence of real numbers such that the series $\sum |\alpha_n|$ converges. Deduce from the previous question a particular solution of the following linear differential equation:

   $$y'' + 4y' + y = \sum_{n=0}^{+\infty} \alpha_n \cos nt.$$

   d) Deduce the solutions of the following linear differential equation:

   $$y'' + 4y' + y = f.$$

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Exercise 1. Consider the following differential equation

$$\text{(E)} \qquad xy''(x) + 2y'(x) + xy(x) = 1 + 3\sin(x)\cos(x)$$

on the interval $I = (0, +\infty)$.

1. Set $y(x) = u(x)v(x)$, with u and v two real valued functions on I of class C^2, and show that the equation (E) can be written as

$$\text{(E')} \qquad M(x)u''(x) + N(x)u'(x) + P(x)u(x) = 1 + 3\sin(x)\cos(x).$$

2. Find on the interval I a function v so that $N = 0$. Deduce the values of $M(x)$ and $P(x)$ for x in I.
3. Solve (E') in that case and deduce the general solution of (E).

Exercise 2. Consider the following initial value problem

$$\text{(P)} \qquad \begin{cases} y'(t)y''(t) - t = 0 \\ y(t_0) = \alpha \\ y'(t_0) = \beta. \end{cases}$$

1. How should you choose (t_0, α, β) to be sure that Problem (P) has a unique solution on some open interval of center t_0?
2. Show that the initial value problem

$$\begin{cases} y'(t)y''(t) - t = 0 \\ y(1) = 2 \\ y'(1) = -1 \end{cases}$$

has a unique solution on an interval of center 1 and determine this solution. Where is this solution valid?

Exercise 3.

1. Give the real form of the solution of the linear differential system

$$\text{(S)} \qquad \begin{cases} x' = -4x + 2y \\ y' = -5x + 2y \\ x(0) = 1, \ y(0) = -1. \end{cases}$$

2. Let ε be a real constant such that $-\frac{1}{2} < \varepsilon < \frac{1}{2}$. We now consider the perturbed system

$$(\text{S}_\varepsilon) \qquad \begin{cases} x' = -4x + 2y \\ y' = (-5 + \varepsilon)x + 2y, \end{cases}$$

a) Find the eigenvalues of the matrix M_ε associated with the linear system (S_ε).
b) Deduce that all solutions $t \mapsto \big(x(t), y(t)\big)$ of System (S_ε) satisfy

$$\lim_{t \to +\infty} x(t) = \lim_{t \to +\infty} y(t) = 0.$$

Exercise 4. Let $E = C\big([0,1]\big)$ be the set of continuous real-valued functions on $[0,1]$. If f belongs to E we set for all $x \in [0,1]$:

$$\Phi(f)(x) = \int_0^1 \left(\int_s^x f(t)\, dt \right) ds.$$

1. Show that if $0 \le x \le y \le 1$ then

$$\big|\Phi(f)(y) - \Phi(f)(x)\big| \le \|f\|_\infty (y - x)$$

   and deduce that for all $x \in [0,1]$ and for all $y \in [0,1]$, one has

$$\big|\Phi(f)(y) - \Phi(f)(x)\big| \le \|f\|_\infty |y - x|.$$

2. Deduce that Φ maps E into E.
3. Show that for all $f \in E$,

$$\big\|\Phi(f)\|_\infty \le \|f\|_\infty.$$

4. Show that Φ is a continuous mapping from $\big(E, \|\cdot\|_\infty\big)$ into $\big(E, \|\cdot\|_\infty\big)$.

SCAN 2 — Test #2 — 1h30 *January 6, 2011*

No document, no calculator, no mobile-phone allowed.

Exercise 1. Show that the series

$$\sum_{n\geq 2} \frac{1}{n^2-1}$$

is convergent and compute the value of its sum.

Exercise 2. We consider the following partial differential equation:

$$\text{(E)} \qquad xy\frac{\partial^2 f}{\partial x^2} + x^2\frac{\partial^2 f}{\partial x \partial y} - y\frac{\partial f}{\partial x} = 0.$$

1. Show that the function f defined on $\mathbb{R}^2$ by $f(x,y) = \mathrm{e}^{x^2-y^2} - 2y^2$ is a solution of (E) on $\mathbb{R}^2$.
2. Does Equation (E) possess solutions on $\mathbb{R}^2$ such that $\forall y \in \mathbb{R}$, $\dfrac{\partial f}{\partial x}(0,y) = 1$?
3. We consider the mapping
$$\Phi(x,y) = \big(x^2 - y^2, y\big).$$
(You might want to introduce the notations $(s,t) = \Phi(x,y)$.)
   a) Find all the points (x_0, y_0) in $\mathbb{R}^2$ such that Φ is a local diffeomorphism of class C^2 in a neighborhood of (x_0, y_0).
   b) We consider the set Ω defined by:
   $$\Omega = \big\{(x,y) \mid x > 0 \text{ and } x^2 > y^2\big\}.$$
   Sketch Ω. Show that Φ can be used to solve Equation (E) on the set Ω. What is the image of Ω by Φ?
   c) Use the new variables $(s,t) = \Phi(x,y)$ to solve Equation (E) on Ω.
4. We now consider $\overline{\Omega} = \big\{(x,y) \in \mathbb{R}^2 \mid x \geq 0 \text{ and } x^2 \geq y^2\big\}$ and f a solution of (E) on Ω still valid on $\overline{\Omega}$.
We remind you that the directional derivative $\partial_u f(x_0,y_0)$ of f in the direction of the vector u at (x_0,y_0) is given by the dot product of the gradient vector of f at (x_0,y_0) and u.
   a) Let $x > 0$. Explain why $n_+ = (1,-1)$ is a normal vector to the boundary of $\overline{\Omega}$ at (x,x), and $n_- = (1,1)$ is a normal vector to the boundary of $\overline{\Omega}$ at $(x,-x)$.
   b) Express the normal derivative of f at (x,x) and at $(x,-x)$, that is, the directional derivatives in a direction normal to the boundary of $\overline{\Omega}$.
   c) Does the solution $f(x,y) = \mathrm{e}^{x^2-y^2} - 2y^2$ have null normal derivatives on the boundary (x,x) of $\overline{\Omega}$? What is the value of the normal derivative of f on the boundary $(x,-x)$ of $\overline{\Omega}$?

Exercise 3. Let f be a real valued function of class C^2 on $\mathbb{R}$. We consider the equation

$$(*) \qquad xz + yf'(z) = 0.$$

1. In this question and in this question only we study the special case $f(z) = \sin z$.
   a) Prove that Equation $(*)$ uniquely determines z as a function φ of (x, y) in some neighborhood of $(1, 0)$, with $\varphi(1, 0) = 0$.
   b) Say why φ is a function of class C^2.
   c) Determine the second order Taylor expansion of φ in a neighborhood of $(1, 0)$.
2. In the general case:
   a) When does Equation $(*)$ uniquely determine z as a function of (x, y) in the neighborhood of $(x_0, y_0, z_0) \in \mathbb{R}^3$?
   We know assume that these conditions are fulfilled and we denote by φ this function, i.e., $z = \varphi(x, y)$.
   b) Say why φ is a function of class C^1 on some open neighborhood U of (x_0, y_0) and compute $\dfrac{\partial \varphi}{\partial x}$ and $\dfrac{\partial \varphi}{\partial y}$ throughout U. *(Clearly mention the calculation points.)*
   c) For (x, y) in U, we set
   $$F(x, y) = x\frac{z^2}{2} + yf(z)$$
   where $z = \varphi(x, y)$.
   Prove that F is of class C^1 on U and that $\dfrac{\partial F}{\partial x}(x, y) = \dfrac{1}{2}\varphi(x, y)^2$.
   Similarly find a simple expression for $\dfrac{\partial F}{\partial y}$.
   Deduce that F is of class C^2 on U and that
   $$\left(\frac{\partial^2 F}{\partial x \partial y}\right)^2 = \left(\frac{\partial^2 F}{\partial x^2}\right) \times \left(\frac{\partial^2 F}{\partial y^2}\right).$$

No document, no calculator, no mobile-phone allowed.

Exercise 1.

1. Give the radius of convergence R_1 of the power series $\sum_{n=1}^{+\infty} \frac{x^n}{n}$ and give its sum on $(-R_1, R_1)$.
2. Give the radius of convergence R_2 of the power series $\sum_{n=0}^{+\infty} \frac{n}{n+1} x^n$ and give its sum on $(-R_2, R_2)$.
3. Deduce the value of the sum of the series

$$\sum_{n=0}^{+\infty} 2^{n+1} \mathrm{e}^{-nx} \frac{n}{n+1}$$

and specify the domain of validity.

Exercise 2. We consider the sequence of functions $(f_n)_{n\in\mathbb{N}}$ defined on $[0, +\infty)$ by

$$f_n(x) = \frac{x^n}{1 + x^{2n}}.$$

1. Compute the pointwise limit of the sequence of functions $(f_n)_{n\in\mathbb{N}}$ on $[0, +\infty)$.
2. Does the series of functions (f_n) converge uniformly on $[0, +\infty)$? If $a > 0$, does the series of functions (f_n) converge uniformly on $[0, a]$? on $[a, +\infty)$? (Discuss according to the value of a).

Exercise 3. The goal of this exercise is to compute the sum of the series

$$S = \sum_{n=1}^{+\infty} \frac{\sin n}{n}.$$

1. Let $x \in (-1, 1)$. Prove that the series

$$\sum_{n=0}^{+\infty} x^n \mathrm{e}^{inx}$$

converges, and compute the value of its sum.

2. Deduce the values of the sums

$$\sum_{n=0}^{+\infty} x^n \cos nx, \qquad \sum_{n=0}^{+\infty} x^n \sin nx.$$

Prove that the corresponding series of functions converge uniformly on $[-a, a]$ for all $a \in (0, 1)$.

3. For $n \in \mathbb{N}^*$ and $x \in \mathbb{R}$ we set

$$u_n(x) = \frac{x^n \sin nx}{n},$$
$$f_n(x) = \sum_{k=1}^{n} u_k(x),$$

and we consider the series of functions

$$f = \sum_{n=1}^{+\infty} u_n.$$

a) Prove that the series of functions f converges on $(-1, 1)$.

b) Prove that f is of class C^1 on $(-1, 1)$ and express f' in terms of usual functions on $(-1, 1)$.

c) A computer algebra system yields:

$$> \int -\frac{x^2 - \cos(x)x - \sin(x)}{x^2 - 2\cos(x)x + 1}\, \mathrm{d}x$$

$$-\frac{x}{2} - \mathrm{ArcTan}\left[\frac{(x+1)\mathrm{Tan}\left[\frac{x}{2}\right]}{x-1}\right]$$

Deduce that for all $x \in (-1, 1)$,

$$f(x) = -\frac{x}{2} - \arctan\left(\frac{(x+1)\tan(x/2)}{x-1}\right).$$

4. a) Let $N \in \mathbb{N}$, $N \geq 2$ and $x \in [0, 1]$. Prove the following equalities:

$$f_N(x) = \sum_{n=1}^{N} \frac{x^n}{n} \sin(nx) = \sum_{n=1}^{N} \frac{x^n}{n} \big(B_n(x) - B_{n-1}(x)\big) = \frac{x^N}{N} B_N(x) + \sum_{n=1}^{N-1} \left(\frac{x^n}{n} - \frac{x^{n+1}}{n+1}\right) B_n(x),$$

where

$$B_n(x) = \sum_{k=0}^{n} \sin(kx).$$

b) Show that the series of functions of x

$$\sum_{n=1}^{+\infty} \left(\frac{x^n}{n} - \frac{x^{n+1}}{n+1}\right)$$

converges normally on $[0, 1]$.

c) It can be shown, and you may use this result without any justification, that for all $x \in (0, 1]$,

$$B_n(x) = \frac{\sin\left(\frac{n}{2}x\right)\sin\left(\frac{n+1}{2}x\right)}{\sin\frac{x}{2}}.$$

Show that the series of functions of x

$$\sum_{n=1}^{+\infty} \left(\frac{x^n}{n} - \frac{x^{n+1}}{n+1}\right) B_n(x)$$

converges normally on $[\pi/4, 1]$.

d) Prove that the sequence of functions $(h_n)_{n\geq 1}$ defined on $[\pi/4, 1]$ by

$$h_n(x) = \frac{x^n}{n} B_n(x)$$

converges uniformly to the zero function on $[\pi/4, 1]$.

e) Deduce that f is continuous on $[\pi/4, 1]$ and deduce that

$$S = \sum_{n=1}^{+\infty} \frac{\sin n}{n} = \frac{\pi - 1}{2}.$$

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SCAN 2 — Test #4 — 1h30 *May 19, 2011*

No document, no calculator, no mobile-phone allowed.

Exercise 1. We consider the mapping φ from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by

$$\varphi(x,y) = \big(x\mathrm{e}^{-y}, x^2+y\big).$$

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a function of class C^2 on $\mathbb{R}^2$ and set $f = F \circ \varphi$. Find all points (x_0, y_0) in $\mathbb{R}^2$ such that φ is a local C^2-diffeomorphism in a neighborhood of (x_0, y_0), and compute $\dfrac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$ in terms of the partial derivatives of F.

Exercise 2. Find the position of the local extreme values on $\mathbb{R}^2$ of the function f defined by

$$f(x,y) = 12xy - 2xy^2 - x^3,$$

and specify it they correspond to local minimum values or local maximum values.

Exercise 3. We consider the quadratic form q on $\mathbb{R}^3$ defined by

$$q(x,y,z) = 4x^2 + y^2 + z^2 - 6yz.$$

We denote by $\mathscr{B}$ the standard basis of $\mathbb{R}^3$.

1. Write the matrix $B = [q]_{\mathscr{B}}$ of q in the basis $\mathscr{B}$.
2. What is the signature of q? Find a basis $\mathscr{C}$ of $\mathbb{R}^3$ that is orthonormal with respect to the standard dot product of $\mathbb{R}^3$, such that the matrix $C = [q]_{\mathscr{C}}$ of q in the basis $\mathscr{C}$ is diagonal. You will explicit the change of basis matrix $P = [\mathscr{C}]_{\mathscr{B}}$ from the basis $\mathscr{B}$ to the basis $\mathscr{C}$ and give a relation between B, C and P.
3. We consider the subspace F of $\mathbb{R}^3$ given by

$$F = \big\{(x,y,z) \in \mathbb{R}^3 \;\big|\; y + z = 0\big\}.$$

   a) Find two vectors u and v of $\mathscr{C}$ that form a basis of F, and prove that for all $\lambda, \mu \in \mathbb{R}$,

$$q(\lambda u + \mu v) = 4\lambda^2 + 4\mu^2.$$

   b) We consider the surface (Σ) of $\mathbb{R}^3$ defined by

$$(\Sigma) = \big\{(x,y,z) \in \mathbb{R}^3 \;\big|\; q(x,y,z) = 1\big\}.$$

   What is the nature of the curve $(\Sigma) \cap F$? Plot $(\Sigma) \cap F$ in $(O; u, v)$.

Please turn over

Exercise 4. We consider the sequence $(b_n)_{n\in\mathbb{N}^*}$ given by

$$b_n = \begin{cases} 0 & \text{if } n = 2 \\ \dfrac{8}{\pi} \cdot \dfrac{(-1)^n - 1}{n\big(n^2-4\big)} & \text{if } n \neq 2. \end{cases}$$

Consider the function f defined on $\mathbb{R}$ by

$$f(x) = \sum_{n=1}^{+\infty} b_n \sin(nx).$$

1. Show that f is continuous on $\mathbb{R}$, periodic of period 2π and odd.
2. Show that for all $x \in [0, \pi]$, $f(x) = 1 - \cos(2x)$.
3. Sketch the graph of f on $[-2\pi, 2\pi]$.
4. For $n \in \mathbb{N}^*$ we consider the following differential equation:

(E$_n$) $$y'(x) + y(x) = b_n \sin(nx).$$

a) For $n \in \mathbb{N}^*$, check that a particular solution on $\mathbb{R}$ of the differential equation (E$_n$) is

$$y_n(x) = A_n \cos(nx) + B_n \sin(nx)$$

where

$$A_n = -\frac{nb_n}{n^2+1}, \qquad B_n = \frac{b_n}{n^2+1}.$$

b) Prove that for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}^*$,

$$\big|y_n(x)\big| \leq 2|b_n|, \qquad \big|y_n'(x)\big| \leq 2|b_n|.$$

c) We consider the series of functions s_P defined by

$$s_P(x) = \sum_{n=1}^{+\infty} y_n(x) = \sum_{n=1}^{+\infty} A_n \cos(nx) + B_n \sin(nx).$$

Prove that s_P is of class C^1 on $\mathbb{R}$ and that it can be differentiated term by term on $\mathbb{R}$.

d) Deduce that s_P is a particular solution of the differential equation

$(*)$ $$y' + y = f$$

on $\mathbb{R}$, and deduce the general form of the solutions of Equation $(*)$ on $\mathbb{R}$.

SCAN 2 — Math Test #1— 1h30 *November 10, 2011*

No document, no calculator, no mobile-phone allowed.

Exercise 1.

1. Let $k \in \mathbb{R}_+^*$ and consider the differential equation:

$$(E_k) \qquad y''(t) + 2ky'(t) + y(t) = 0.$$

   a) Solve the differential equation (E_k) on $\mathbb{R}$ (give the real solutions).
   b) Prove that for all $k \in \mathbb{R}_+^*$, every solution of (E_k) approaches 0 as $t \to +\infty$.
   c) Solve the differential equation
$$y''(t) + 4y'(t) + y(t) = \sin t$$
   on $\mathbb{R}$ (give the real solutions).

2. We consider the differential equation

$$(E') \qquad x^2 z''(x) + 3xz'(x) + z(x) = 0.$$

   a) Is the set of real solutions on $\mathbb{R}_+^*$ of the differential equation (E') a real vector space? Can you write the characteristic equation of the differential equation (E')?
   b) The goal is now to solve (E') on $\mathbb{R}_+^*$.
      i) For $t \in \mathbb{R}$, set $y(t) = z\big(\mathrm{e}^t\big)$. Show that z is a solutions of (E') on $\mathbb{R}_+^*$ if and only if y is solution of (E_k) on $\mathbb{R}$ for a value of $k \in \mathbb{R}_+^*$ you will determine.
      ii) Deduce all the solutions of (E') on $\mathbb{R}_+^*$.

Exercise 2. In this exercise we consider the real vector space $E = C\big([0,1]\big)$ that consists of all continuous real-valued functions on $[0,1]$. We recall that

$$\forall \alpha, \beta \in \mathbb{R}, \ |\cos\alpha - \cos\beta| \leq |\alpha - \beta|$$

and you may use this result without any justification.
Let $a \in \mathbb{R}$. We define a mapping Φ on E thus: If $f \in E$ we set $\Phi(f) = g$, where

$$\forall x \in [0,1], \ g(x) = a + \int_0^x \big(1 + t\cos f(t)\big)\,\mathrm{d}t.$$

1. Show that $\Phi(E) \subset E$. Is Φ an endomorphism of E?

2. Show that:
$$\forall f_1, f_2 \in E, \ \big\|\Phi(f_1) - \Phi(f_2)\big\|_\infty \leq \frac{1}{2}\|f_1 - f_1\|_\infty.$$

3. Would you say that Φ is a continuous mapping from $\big(E, \|\cdot\|_\infty\big)$ to $\big(E, \|\cdot\|_\infty\big)$? Justify your answer.

4. We consider the following differential problem:

$$(P) \qquad \begin{cases} y \text{ is of class } C^1 \text{ on } [0,1], \\ y'(t) = 1 + t\cos y(t), \ t \in [0,1], \\ y(0) = a. \end{cases}$$

   a) Prove that if y is a solution of Problem (P), then $\Phi(y) = y$.
   b) Deduce that Problem (P) possesses at most one solution.

Exercise 3. This exercise consists of 4 independent questions.

1. Consider the curve $\mathscr{C}$ of $\mathbb{R}^2$ given by
$$\mathscr{C} : x = y^2,$$
and the point $m_0(4, -2)$ of $\mathscr{C}$.

   a) Give a unit vector $\vec{t}$ tangent to $\mathscr{C}$ at the point m_0.

   b) Give a unit vector $\vec{n}$ normal to $\mathscr{C}$ at the point m_0.

   c) Compute the directional derivatives of the function f defined on $\mathbb{R}^2$ by $f(x, y) = x^2 y$ at the point m_0 in the direction $\vec{t}$ and in the direction $\vec{n}$.

2. We assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is a function of class C^1. We denote by $\dfrac{\partial f}{\partial x}(x, y)$ and $\dfrac{\partial f}{\partial y}(x, y)$ the partial derivatives of f with respect to the first and second variable at (x, y) respectively. For $x \in \mathbb{R}$ we set
$$g(x) = f\Big(f(x, x^2), f(x, x)\Big).$$
Compute $g'(x)$, for $x \in \mathbb{R}$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class C^2 and consider
$$\begin{array}{rccc} g : & \mathbb{R} \times \mathbb{R}^* & \longrightarrow & \mathbb{R} \\ & (x, y) & \longmapsto & f\left(\dfrac{x^2}{y}\right). \end{array}$$
Compute $\dfrac{\partial^2 g}{\partial x \partial y}$ at a point $(x, y) \in \mathbb{R} \times \mathbb{R}^*$.

4. Find:

   a) All real-valued functions of class C^2 on $\mathbb{R}^2$ such that
$$\forall (x, y) \in \mathbb{R}^2,\ \frac{\partial^2 f}{\partial x \partial y}(x, y) = 0.$$

   b) All real-valued functions of class C^1 on $\mathbb{R}^2$ such that
$$\forall (x, y) \in \mathbb{R}^2,\ \frac{\partial f}{\partial x}(x, y) + x f(x, y) = 0.$$

SCAN 2 — Math Test #2— 1h30 *January 5, 2012*

No document, no calculator, no mobile-phone allowed.

Exercise 1. Consider the following partial differential equation

$$(*) \qquad 2x^2\frac{\partial^2 f}{\partial x^2} + 3xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = 0,$$

where f is an unknown real-valued function of class C^2 on $\Omega = \mathbb{R}_+^* \times \mathbb{R}_+^*$.

1. Show that the mapping f defined by $f(x,y) = x - \dfrac{y^2}{x}$ is a solution of $(*)$ on Ω.
2. Show that the mapping Φ defined on Ω by $\Phi(x,y) = \left(\dfrac{x}{y}, \dfrac{y^2}{x}\right)$ defines a local C^2-diffeomorphism in the neighborhood of any point of Ω. We denote by (u,v) the new coordinates induced by Φ, i.e., $u = x/y$ and $v = y^2/x$.
3. Sketch on the same figure the part in Ω of the curves $(C_{u,k})$ of equation $\dfrac{x}{y} = k$ and $(C_{v,k})$ of equation $\dfrac{y^2}{x} = k$ for $k = \dfrac{1}{2}$, $k = 1$ and $k = 2$.
4. Graphically justify that $\Phi : \Omega \to \Omega$ defines a C^2-diffeomorphism.
5. Explicit Φ^{-1}. Compute the Jacobian matrix J_Φ of Φ and the Jacobian $J_{\Phi^{-1}}$ of Φ^{-1}. Give the relationship between these matrices, and check it in this particular case.
6. Set $f(x,y) = g(u,v)$ on Ω. Show that Equation $(*)$ in f is equivalent to the following equation in g:
$$\frac{\partial^2 g}{\partial u \partial v}(u,v) - \frac{1}{v}\frac{\partial g}{\partial u}(u,v) = 0$$
and solve it on Ω.
7. Deduce that any solution of $(*)$ can be written as $f(x,y) = xA\left(\dfrac{x}{y}\right) + B\left(\dfrac{y^2}{x}\right)$ where A and B are functions of class C^2 on $\mathbb{R}_+^*$.
8. Give a tangent vector $\vec{t}$ to the curve of equation $x = y^2$ and find the solutions of $(*)$ on Ω that are null on the line $x = y$ and such that the directional derivative $\partial_{\vec{t}} f$ vanishes throughout the curve $x = y^2$.

Exercise 2. Let (Σ) be the set of points $M(x,y,z)$ such that $x\mathrm{e}^y + y\mathrm{e}^z + z\mathrm{e}^x = 0$.

1. Show that in the neighborhood of $O(0,0,0)$, (Σ) possesses a local representation of the form $z = \varphi(x,y)$.
2. Explain why φ is of class C^∞ in the neighborhood of O. Give the value of $\varphi(0,0)$. Give the Taylor expansion of order 2 of φ at $(0,0)$.
3. Compute the value of the limit $\displaystyle\lim_{x\to 0} \frac{\varphi(x,-x)}{x^2}$.

Exercise 3.

1. Show that the series $\displaystyle\sum_{n\geq 1} \frac{1}{n(n+1)(n+2)}$ converges and compute the value of its sum S. How many terms are sufficient to obtain an approximation of S with error not exceeding 10^{-3}?
2. Show that the series $\displaystyle\sum_{n\geq 1} \frac{(-1)^n}{n(n+1)(n+2)}$ converges and compute the value of its sum S. How many terms are sufficient to obtain an approximation of S with error not exceeding 10^{-3}? We recall that $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} = \ln 2$.
3. Let $\theta \in \mathbb{R}$. Show that the series $\displaystyle\sum_{n\geq 0} \frac{\cos(n\theta)}{3^n}$ converges and compute the value of its sum S. How many terms are sufficient to obtain an approximation of S with error not exceeding 10^{-3}?

SCAN 2 — Math Test #3— 1h30 *March 15, 2012*

No document, no calculator, no mobile-phone allowed.

Exercise 1. We consider the sequence of functions $(f_n)_{n\in\mathbb{N}}$ on $\mathbb{R}_+$ defined by

$$f_n(x) = \mathrm{e}^{-nx}\cos(nx).$$

1. Find the pointwise limit of the sequence of functions $(f_n)_n$ on $\mathbb{R}_+$.
2. a) Does $(f_n)_n$ converge uniformly on $\mathbb{R}_+$?
   b) Does $(f_n)_n$ converge uniformly on $(0, +\infty)$?
   c) Does $(f_n)_n$ converge uniformly on $[a, b]$ with $0 < a < b$?

Exercise 2. We recall the following results, and you may use them without any justification:

- $\forall t > 0$, $\arctan t + \arctan \dfrac{1}{t} = \dfrac{\pi}{2}$;
- $\forall t \in \mathbb{R}$, $|\arctan t| \leq |t|$;
- $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

We consider the series of functions

$$f(x) = \sum_{n=1}^{+\infty} \frac{\arctan(nx)}{n^2}.$$

1. Show that f is a continuous, bounded and odd function on $\mathbb{R}$.
2. What are the variations of f on $\mathbb{R}$?
3. Prove that f is of class C^1 on $\mathbb{R}_+^*$ and give an expression of f' on $\mathbb{R}_+^*$.
4. For $x \in \mathbb{R}_+^*$, we set $\varphi(x) = \dfrac{\pi^3}{12} - f(x)$.
   a) Show that there exists $K \in \mathbb{R}$ such that $\forall x > 0$, $|\varphi(x)| \leq \dfrac{K}{x}$.
   b) Deduce that f has a finite limit at $+\infty$ and determine the value of this limit.
5. Use a comparison with an integral to prove that for all $x > 0$, $f'(x) \geq \dfrac{1}{2}\ln\left(\dfrac{1+x^2}{x^2}\right)$. Deduce the value of $\displaystyle\lim_{x\to 0^+} f'(x)$.
6. Sketch the graph of f on $\mathbb{R}$ (you will show all the elements that have been obtained in the previous questions on your graph).

Exercise 3. We consider the power series $g(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

1. Prove that g is well-defined and continuous on $[-1, 1]$.
2. a) Prove that
$$\forall x \in (-1, 1), \ \arctan(x) = g(x).$$
   b) Is the previous equality still valid on $[-1, 1]$? Justify your answer.
3. We now consider the power series
$$f(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)(2n+2)} x^{2n+1}$$
and the differential equation
$$\text{(E)} \qquad xy'(x) + y(x) = \arctan(x).$$
   a) Determine the radius of convergence R of the power series f.
   b) Prove that f is a solution of (E) on $(-R, R)$.
   c) Give an expression of f on $(-R, R)$ in terms of usual functions. *Hint: You may either use a partial fraction decomposition of* $\frac{1}{(2n+1)(2n+2)}$ *or solve the differential equation* (E) *or use another method of your choice. We recall that for all* $x \in (-1, 1)$, $\ln(1 + x) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^n}{n}$.

SCAN 2 — Math Test #4— 1h30 *May 3, 2012*

No document, no calculator, no mobile-phone allowed.

Exercise 1. Let $\lambda > 0$. We consider the differential equation

$$(E_\lambda) \qquad xy''(x) + 2y'(x) + \lambda xy(x) = 0.$$

The goal is to find the solutions of (E_λ) that possess a power series expansion.

1. Let $f(x) = \sum_{n=0}^{+\infty} a_n x^n$ be a power series with radius R that is a solution of (E_λ). Find the relations between the a_n's and deduce that

$$\forall p \in \mathbb{N}, \quad \begin{cases} a_{2p} = \dfrac{(-\lambda)^p}{(2p+1)!} a_0 \\ a_{2p+1} = 0. \end{cases}$$

2. Deduce the value of R and express the solutions of (E_λ) on $(-R, R)$ that possess a power series expansion, in terms of usual functions.

Exercise 2. Questions 2 and 3 of this exercise are independent from each other.

1. Let E be a real vector space and let $\langle \cdot | \cdot \rangle$ be an inner product on E, and $\| \cdot \|$ the associated norm. Say why for all $u, v \in E \setminus \{0\}$, $\dfrac{\langle u|v \rangle}{\|u\|\|v\|} \in [-1, 1]$.

   Definition. The number $\theta = \arccos\left(\dfrac{\langle u|v \rangle}{\|u\|\|v\|}\right)$ is called the *non-oriented angle between u and v with respect to* $\langle \cdot | \cdot \rangle$.

2. In this question only we consider $E = \mathbb{R}^4$. We consider the vectors $u = (0, 1, 0, 1)$ and $v = (1, 1, 1, 1)$ of E.
   a) Compute the non-oriented angle between u and v with respect to the standard dot product of $E = \mathbb{R}^4$.
   b) We now consider the following quadratic form on E:

   $$q(x, y, z, t) = x^2 + 2xy + 2y^2 + z^2 + t^2,$$

   and we denote by φ the associated polar form.
      i) Show that φ is an inner product on E.
      ii) Write the matrix $[q]_{\text{std}}$ of the quadratic form q in the standard basis of E.
      iii) Compute the cosine of the non-oriented angle between u and v with respect to φ.
      iv) Compute the orthogonal projection (with respect to φ) of the vector $w = (1, 0, 0, 0)$ on the subspace $F = \text{Span}\{u, v\}$.

3. In this question only we consider the vector space $E = \mathbb{R}_3[X]$, that consists of polynomials of degree at most 3 with real coefficients, together with the symmetric bilinear form φ on E given by

   $$\varphi(P, Q) = P(0)Q(0) + \int_{-1}^{1} P'(t)Q'(t)\,\mathrm{d}t.$$

   We recall that the standard basis of E is $\mathscr{C} = (1, X, X^2, X^3)$. We also consider the following vectors of E:

   $$S_1 = X^3, \qquad S_2 = 5X^3 - 9X, \qquad S_3 = 3X^2 - 8, \qquad S_4 = X^2 + 1.$$

   a) Show that φ is an inner product on E and explicit the matrix $A = [\varphi]_{\mathscr{C}}$. Is the basis $\mathscr{C}$ an orthogonal basis of E?
   b) What is the cosine of the non-oriented angle between X and X^3 with respect to φ?

c) You are given that $\mathscr{B} = (S_1, S_2, S_3, S_4)$ is a basis of E. Explicit the change of bases matrix $P = [\mathscr{B}]_{\mathscr{C}}$ from the basis $\mathscr{C}$ to the basis $\mathscr{B}$. Use the change of basis formula to show that

$$[\varphi]_{\mathscr{B}} = \begin{pmatrix} 18/5 & 0 & 0 & 0 \\ 0 & 72 & 0 & 0 \\ 0 & 0 & 88 & 0 \\ 0 & 0 & 0 & 11/3 \end{pmatrix}.$$

What can you deduce about $\mathscr{B}$?

d) We now consider the vector $Q = 1$ of E. Compute the orthogonal projection of the vector Q on the subspace $F = \text{Span}\{S_1, S_2, S_3\}$.

Exercise 3. Let E be a real vector space and let $\langle\cdot|\cdot\rangle$ be an inner product on E. We denote by $\|\cdot\|$ the associated Euclidean norm. Let f be an endomorphism of E such that

$$\forall u \in E, \ \|f(u)\| = \|u\|.$$

1. Prove that for all $u, v \in E$, $\langle f(u)|f(v)\rangle = \langle u|v\rangle$. Hint: Compute $\|f(u+v)\|^2$.
2. If E is finite-dimensional and $\mathscr{B}$ is an orthonormal basis of E, show that $A = [f]_{\mathscr{B}}$ is an orthogonal matrix (i.e., ${}^tAA = \text{Id}$).

Exercise 4. Let $a > 0$. Consider the function f, periodic on $\mathbb{R}$ of period $2a$, defined on $(-a, a]$ by

$$f(x) = \begin{cases} -1 & \text{if } x \in (-a, 0) \\ 0 & \text{if } x = 0 \text{ or } x = a \\ 1 & \text{if } x \in (0, a). \end{cases}$$

1. Plot the graph of f on $[-2a, 4a]$.
2. Write the Fourier series $S(f)$ of f.
3. Do we have $S(f) = f$? Justify your answer.
4. Is the series $S(f)$ uniformly convergent on $\mathbb{R}$?