Interrogations Écrites
SCAN - Deuxième année

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SCAN **Mathematics (test 1)** **October 25, 2007**

1h30

Exercise 1: (~ 7 points)

1) Find the real solutions of the following differential system (S): $\begin{cases} x_1'(t) = x_2(t) \\ x_2'(t) = x_3(t) \\ x_3'(t) = -2x_1(t) - 4x_2(t) - 3x_3(t) \end{cases}$.

Express them without any complex number.

2) Deduce the solution of the following initial value problem:

$$\begin{cases} y'''(t) + 3y''(t) + 4y'(t) + 2y(t) = 0 \\ y(0) = 1, y'(0) = 0, y''(0) = 0 \end{cases}$$

Express your answer without any complex number.

3) Find the real solutions of $y'''(t) + 3y''(t) + 4y'(t) + 2y(t) = t^2$

Exercise 2: (~ 8.5 points)

A factoring method for solving linear differential equations.

If I is an interval of R, if a is a continuous function from I to R and if y is a C[1] function from I to R, define $(D + a(x))\, y(x)$ as being $y'(x) + a(x)y(x)$.

1) Solve the following differential equations:

a) $y'(x) + x\, y(x) = x$

b) $y'(x) + \frac{1}{x} y(x) = \lambda e^{-x^2/2} + 1$ where λ is a given real constant.

2) Consider the following differential equation:

(E) : $x^2 y''(x) + (x^3 + x)y'(x) + (x^2 - 1)y(x) = x^3$

a) Where is it possible to solve (E)?

b) Suppose that I is an interval in which (E) can be solved.
Prove that (E) can be rewritten in I as:

$$(D + x)(D + \frac{1}{x})y(x) = x$$

c) Use 1) to deduce the solutions of (E) in I.

3) More generally, consider the following differential equation:

(GE) : $y''(x) + p_1(x)y'(x) + p_0(x)y(x) = f(x)$

where p_0, p_1 and f are given real-valued continuous functions in an interval I.

Show that if (GE) can be rewritten in I as $(D + a_1(x))(D + a_2(x))y(x) = f(x)$ for some functions a_1 and a_2, then a_2 must be at least C[1] and $\begin{cases} a_1(x) = p_1(x) - a_2(x) & (1) \\ a_2'(x) = a_2^2(x) - p_1(x)a_2(x) + p_0(x) & (2) \end{cases}$.

Remark: Unfortunately (2) *is a Ricatti first order differential equation of an unknown function a_2 that can be easily solved when one particular solution is known, which is not always the case….*

Exercise 3: (~ 4 points)

Show the convergence and find the sum of the following series:

1) $\sum_{n \geq 0} \frac{1}{n^2 + 4n + 3}$

2) $\sum_{n \geq 0} \frac{\cos(n\theta)}{2^n}$ where θ is a given real number.

SCAN | **Test of Mathematics (2)** **(1 h 30)** | **December 13, 2007**

Clearly state the theorems you use.

Numerical series

Exercise 1

Test for convergence the following series:

1) $\sum \frac{3^n}{n!}$ 2) $\sum (-1)^n \frac{\cos(2n\theta)}{2^n}$ $(\theta \in \mathbb{R})$ 3) $\sum \frac{(-1)^n}{\sqrt{n}} \cos(\frac{1}{n})$

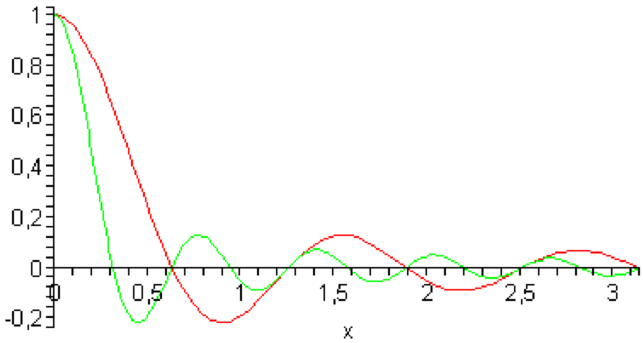
4) $\sum (1+\frac{1}{n})^{n^2}$ 5) $\sum (1-\frac{1}{n})^{n^2}$ 6) $\sum (e^{\frac{a}{n}} - \sqrt{1+\frac{1}{n}})$ according to $a \in \mathbb{R}$.

Sequences of functions

Exercise 2

For $n \in \mathbb{N}^*$, put $f_n(x) = \begin{cases} \frac{\sin(nx)}{nx} & \text{for } x \in]0,\pi] \\ 1 & \text{for } x = 0 \end{cases}$ (*On the figure below you can see* f_5 *and* f_{10})

Study the pointwise and uniform convergence of the <u>sequence</u> of functions $\{f_n\}_{n \in N^*}$ on $[0, \pi]$, on $]0, \pi]$ and on $[a, \pi]$ with $0 < a < \pi$.



<u>**Series of functions**</u>

Exercise 3

The following two sums are given and can be used without proof: $\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n}=\ln 2$ and $\sum_{n=1}^{+\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$

Put $f_n(x)=\frac{x}{n(1+nx)}$ and $f(x)=\sum_{n=1}^{+\infty}f_n(x)$ for $x\geq 0$.

1) Say why f is well defined if $x\in[0,+\infty[$.

2) Calculate $f(0)$, $f(1)$ and $f(2)$

(*hint*: for f(2) you may decompose $f_n(2)$ and write $\frac{1}{n}=\frac{2}{2n}$.)

3) Show that the sequence of functions $\{f_n\}$ is uniformly convergent on $[0, +\infty[$.

4) Show that the series of functions $\sum f_n$ is uniformly convergent on $[0, +\infty[$.

5) Prove that f is continuous on $[0, +\infty[$.

6) Prove that the series $\sum\frac{1}{n^2}(1-\frac{\ln(n+1)}{n})$ converges and that $\int_0^1 f(x)dx = \sum_{n=1}^{+\infty}\frac{1}{n^2}(1-\frac{\ln(n+1)}{n})$

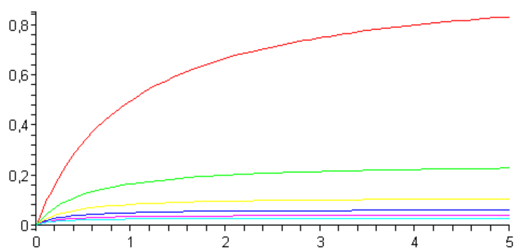
7) a) Show that f is C^1 on $]0,+\infty[$ and find the variations of f on $]0,+\infty[$.
Is f' necessary to find the variations of f?

b) Calculate $f'(1)$.

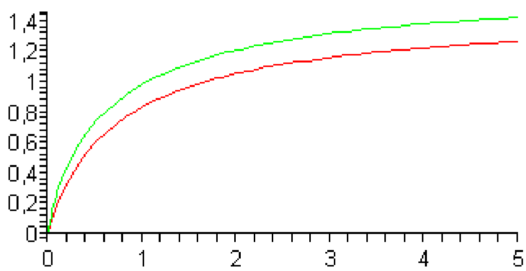
8) a) After having checked that $\forall x>0, \forall N\in N^*, \frac{f(x)}{x}\geq\sum_{n=1}^{N}\frac{1}{n(1+nx)}$

Prove that $\forall x>0, \forall N\in N^*, N\leq\frac{1}{x}\Rightarrow\frac{f(x)}{x}\geq\sum_{n=1}^{N}\frac{1}{2n}$.

b) Deduce that the curve of f has a vertical tangent on the right at 0.



Representation of f_n on [0,5] for n =1,2, … 6 given by Maple



Representation of 2 partial sums $S_n=\sum_{p=1}^{n}f_p$ on [0,5] for n=5 and 50 given by Maple.

SCAN **Test of mathematics (3)** **March 27, 2008**

(1h30)

No documents, no calculators, no mobile phones allowed

Exercise 1 (~ 6 – 7 points)

We can use without proof the following values: *If* $I_n = \int_{-\pi}^{\pi} \cos^n t\, dt$ *for* $n \in N$ *then*

$$I_0 = 2\pi,\ I_1 = 0, I_2 = \pi,\ I_3 = 0, I_4 = \frac{3\pi}{4},\ I_5 = 0, I_6 = \frac{5\pi}{8},\ I_7 = 0, I_8 = \frac{35\pi}{64},\ I_9 = 0, \ldots$$

1) Prove that the mapping φ defined by $\varphi(f,g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)g(t)\,dt$ defines an inner product on E where $E = \mathcal{C}([-\pi,\pi],\mathrm{R})$, the set of continuous functions from $[-\pi,\pi]$ into R.

Let e_1, e_2, and e_3 be the 3 functions defined by $e_1(t) = 1$, $e_2(t) = \cos t$ and $e_3(t) = \cos^2 t$ for $t \in [-\pi, \pi]$
Call F the subspace of E that is spanned by these 3 functions *i.e.* $F = \{a_1 e_1 + a_2 e_2 + a_3 e_3 \mid a_1, a_2, a_3 \in R\}$
Admit that $\mathcal{B} = (e_1, e_2, e_3)$ is a basis of F and still call φ the restriction of φ to F.

2) Write the matrix A of φ in the basis $\mathcal{B} = (e_1, e_2, e_3)$.

3) Apply the Schmidt process to $\mathcal{B}$ to find an orthogonal basis $\mathcal{B}_1 = (f_1, f_2, f_3)$ of F versus φ and deduce an orthonormal basis $\mathcal{B}_2 = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ of F (versus φ).
What is the matrix A' of φ in $\mathcal{B}_2$? Verify the transition formula.

4) Let f be the function of E defined by $f(t) = \cos^4 t$.
Find the orthogonal projection (versus φ) of f onto F *i.e.* find g in F such that $f - g$ is orthogonal to F (versus φ).

Exercise 2(~ 7 – 8 points)

In this exercise, $R^{3*} = R^3 - \{(0,0,0)\}$.

1) Let λ_1, λ_2 and λ_3 be 3 real numbers such that $\lambda_1 \leq \lambda_2 \leq \lambda_3$

Prove that $\max\limits_{(x,y,z)\in R^{3*}} \dfrac{\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2}{x^2 + y^2 + z^2} = \lambda_3$

2) Consider the quadratic form of R^3 defined by $Q(x,y,z) = 2x^2 + 2y^2 + 2z^2 + 2xy - 2xz - 2yz$.

a) What is the signature of Q.

b) Write Q as an algebraic sum of squares in an orthonormal basis $\mathcal{B}'$ of R^3 for its usual dot product $\bullet$.

c) Use the basis $\mathcal{B}'$ to prove that $\max\limits_{(x,y,z)\in R^{3*}} \dfrac{Q(x,y,z)}{x^2 + y^2 + z^2} = 4$.

3) Put M = $\begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

a) Calculate ${}^tMM$.

b) Deduce from part 2, that $\sup\limits_{X\in R^3 - \{\begin{pmatrix}0\\0\\0\end{pmatrix}\}} \dfrac{\|MX\|_2}{\|X\|_2} = 2$.

c) Which result from the lecture has been proved in this special case?

Exercise 3 (~ 5 - 6 points)

Call $E = \mathcal{C}([0,1],R)$.

Consider the following integral equation $\forall x\in [0,1]$, $y(x) = 1 + \int_0^x t\, y(t)dt$ of an unknown function y in E.

1) Use the fixed point theorem to show that the above integral equation has a unique solution in E.

2) Say how to construct a sequence y_n of functions that uniformly converges to y in [0,1] and such that $\forall x\in [0,1]$, $y_0(x)=0$ and give an upper bound of $\sup\limits_{x\in[0,1]} |y_n(x) - y(x)|$.

3) Prove that $\forall n \in N^*, \forall x \in [0,1], y_n(x) = \sum\limits_{k=0}^{n-1} \dfrac{x^{2k}}{2^k k!}$.

Recognize y.

SCAN **Test of Mathematics (4)** **May 22, 2008**
(1h 30)

No documents, no calculators, no mobile phones allowed

Exercise 1 ~ 3 points

Consider the following change of variables φ defined by $\begin{cases} u = xy^2 \\ v = x^2 y \end{cases}$.

(*We do not ask for the study of this change of variables)*
Let f be a C^2 function in R^2 and put $f(x,y) = F(u,v)$ *i.e.* $f = F o \varphi$

Using the notation $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, ... for the partial derivatives of f and the notation $\frac{\partial F}{\partial u}$, $\frac{\partial F}{\partial v}$, ... for the partial derivatives of F, express $\frac{\partial^2 f}{\partial x \partial y}(x,y)$ in terms of the partial derivatives of F.

Exercise 2 ~ 9-10 points

The 3 parts are independent of each other and can be treated in any order.
Let Ω be an open subset of R^2 and let $f : \Omega \subset R^2 \to R^2$ be a C^2 function. Put $f(x,y) = (u(x,y), v(x,y))$.
One says that f satisfies the Cauchy-Riemann conditions in Ω if

$$\forall (x,y) \in \Omega, \begin{cases} \frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) \\ \frac{\partial u}{\partial y}(x,y) = -\frac{\partial v}{\partial x}(x,y) \end{cases} \text{ or if you prefer } \forall (x,y) \in \Omega, \begin{cases} \partial_1 u(x,y) = \partial_2 v(x,y) \\ \partial_2 u(x,y) = -\partial_1 v(x,y) \end{cases}.$$

The Laplacian Δ of a C^2 real valued function g of 2 variables x and y is $\Delta g = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}$.

1) Show that if f satisfies the Cauchy-Riemann conditions in Ω then $\Delta u = 0$ on Ω.

2) Let $\Phi :]0,+\infty[\to R$ be a C^2 real valued function and put $u(x,y) = x\Phi(\sqrt{x^2+y^2})$.
a) Find the second order differential equation that has to be satisfied by Φ to have $\Delta u = 0$ on $\Omega = R^2 - \{(0,0)\}$ (*you can introduce the variable* $r = \sqrt{x^2+y^2}$).
b) Find all the functions u of the form $u(x,y) = x\Phi(\sqrt{x^2+y^2})$ such that $\Delta u = 0$ on $\Omega = R^2 - \{(0,0)\}$.
c) For each such u, find a C^2 function v on $\Omega = R^2 - \{(0,0)\}$ such that $f = (u,v)$ satisfies the Cauchy-Riemann conditions on $\Omega = R^2 - \{(0,0)\}$.

3) Suppose that $f = (u,v)$ satisfies the Cauchy-Riemann conditions on an open set Ω. (*we no longer consider the special case studied in part 2*)
a) Show that the Jacobian of f is equal to 0 at a point (x,y) of Ω if and only if the Jacobian matrix of f at (x,y) is the zero matrix.
b) Suppose that the Jacobian matrix of f is not zero at a point (x_0, y_0) of Ω, prove that f is locally invertible in the neighborhood of (x_0, y_0) and show that f^{-1} also checks the Cauchy-Riemann conditions.

Exercise 3 ~ 5 points

The area of a triangle sides *x*, *y* and *z* of perimeter 2*p* (*i.e.* *x*+*y*+*z* = 2 *p*) is given by $A(x,y,z)=\sqrt{p(p-x)(p-y)(p-z)}$

For a given parameter 2 *p* find *x*, *y* and *z* to get the largest possible area. (*For the values you find, at least show that you have a local maximum and then try to prove you have an absolute maximum*)

Exercise 4 ~ 3 points

Let $f:\Omega\subset R^2\to R$ be a C^1 function in the open set Ω.

Consider the set Σ of points M of coordinates (*x*,*y*,*z*) such that *f*(*x*,*y*,*z*) = 0.

Suppose that *f*(*a*,*b*,*c*) = 0.

1) Under what conditions are you sure that the equation can be solved in z in the neighborhood of (*a*,*b*,*c*) (*that is to say* that the equation uniquely defines z in the neighborhood of (*a*,*b*,*c*) as a function φ of (*x*,*y*))?

In this case, define $\left(\frac{\partial z}{\partial x}\right)_y$ as being $\frac{\partial\varphi}{\partial x}$ and $\left(\frac{\partial z}{\partial y}\right)_x$ as being $\frac{\partial\varphi}{\partial y}$.

2) Assume that in the neighborhood of (*a*,*b*,*c*), the equation can be solved in *x*, in *y* and also in *z*.

Give the required conditions and prove that $\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y=-1$ at (*a*,*b*,*c*).

3) **Bonus:** State the corresponding result for an equation $f(x_1,x_2,x_3,x_4)=0$.

SCAN **Test of Mathematics (1)** **November 13, 2008**

(1 h 30)

No documents, no calculators, no mobile phones allowed
Clearly state the theorems you use.

Exercise 1 (~5 points)

Test for convergence the following five series:

a) $\sum_{n\geq 0} \frac{e^{-na}}{n!}$ according to a in R b) $\sum_{n\geq 2} (-1)^n \frac{n}{(\ln n)^2}$ c) $\sum_{n\geq 0} (-1)^n \frac{\cos n}{n^2+1}$

d) $\sum_{n\geq 2} (-1)^n \frac{(\ln n)^2}{n}$ e) $\sum_{n\geq 2} \frac{(-1)^n}{\sqrt{n}+(-1)^n}$

Exercise 2 (~2-2,5 points)

Calculate $\sum_{p=2}^{+\infty}\left(\sum_{n=2}^{+\infty}\frac{1}{p^n}\right)$

Exercise 3 (~4,5 -5 points)

Put $f_n(x) = e^{-x^n} = \exp(-x^n)$ for $x \in [0, +\infty[$ and $n \in N^*$.

1) Show that the sequence of functions $\{f_n\}_{n\in N^*}$ is pointwise convergent to a function f on $[0, +\infty[$.
2) Study the uniform convergence of the sequence $\{f_n\}_{n\in N^*}$ on $[a, +\infty[$ according to $a \geq 0$.
3) Is the convergence uniform on $[0,b]$ with $0 < b < 1$?
4) **S**tudy the pointwise convergence of the series of functions $\sum f_n$ on $[0, +\infty[$.

Exercise 4 (~8 points)

The following result can be used without proof $\forall x > 0, Arc\tan x + Arc\tan\frac{1}{x} = \frac{\pi}{2}$

I) For $n \in N^*$ and $a \in R$, put $u_n = \frac{1}{n^2+a^2}$.

Show that the series $\sum u_n$ converges and, comparing series and integral, prove that

$$\forall n \in N^*, \forall a \in R^*, 0 \leq \sum_{p=n+1}^{+\infty} u_p \leq \frac{1}{a} Arc\tan\frac{a}{n}$$

(*Details of the proof are required. Don't just apply a theorem*)

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II) For $n \in N^*$ and $x \in R$, put $f_n(x) = \frac{1}{n^2+x^2}$ and $f(x) = \sum_{n=1}^{+\infty} f_n(x)$.

1) Rapidly say why f is well defined on R and check that f is even.
2) Prove that the series of functions $\sum f_n$ is uniformly convergent on R and deduce that f is continuous on R.
3) Use part **I)** to find $\lim_{x\to+\infty} f(x)$.
4) Without trying to differentiate f, give its variations on R.
5) Prove that f is C^1 and can be differentiated term by term on R.

SCAN **Test of Mathematics (2)** **December 18, 2008**

(1 h 30)

No documents, no calculators, no mobile phones allowed
Clearly state the theorems you use.

Exercise 1 (~2 points)

Give the radius of convergence and find the sum on its open interval of convergence of the following power series: $\sum_{n=0}^{+\infty}(-1)^n n x^n$. Deduce the value of the following sum: $s = \sum_{n=0}^{+\infty}(-1)^n \frac{n}{2^n}$.

Exercise 2 (~12-13 points)

Part I results can be used without proof to answer Part II.

Part I (~7-8 points)

Consider the **odd** periodic function f of **period 2π** such that $f(x) = x(\pi - x)$ in $[0,\pi]$.

1) Plot its curve on $[-3\pi,3\pi]$. (*Don't forget that f is odd)*

2) Calculate the Fourier series of f.

3) Deduce that for all x in $\mathbb{R}$, $f(x) = \frac{8}{\pi}\sum_{k=0}^{+\infty}\frac{1}{(2k+1)^3}\sin((2k+1)x)$. Clearly justify your answer.

4) Deduce the value of the following sums $s_1 = \sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k+1)^3}$ and $s_2 = \sum_{k=0}^{+\infty}\frac{1}{(2k+1)^6}$.

Part II (~5-6 points)

In this part, c is a given positive real number.

Suppose that $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers such that

$$\exists K \in \mathbb{R}^+, \exists \alpha > 1, \forall n \in \mathbb{N}^*, |A_n| \le \frac{K}{n^\alpha}.$$

For $x \in \mathbb{R}$ and $t \in \mathbb{R}$, put $F(x,t) = \sum_{n=1}^{+\infty} A_n \cos(nct)\sin(nx) = \sum_{n=1}^{+\infty} u_n(x,t)$ with $u_n(x,t) = A_n \cos(nct)\sin(nx)$

1) Prove that the series defining F is convergent for all $x \in \mathbb{R}$ and $t \in \mathbb{R}$.

For a given t, we will consider the function of x defined by $g_t(x) = F(x,t)$.

For a given x, we may also consider the function of t defined by $h_x(t) = F(x,t)$.

2) Prove that g_t is continuous on $\mathbb{R}$ (t is fixed). What do you get for h_x?

3) Prove that if $\alpha > 3$, g_t is C^2 on $\mathbb{R}$ and that it can be twice differentiated term by term.
What do you get for h_x?

4) Deduce that if $\alpha > 3$, F is a solution on $[0,\pi]\times\mathbb{R}$ of the following problem:

$$\begin{cases} \frac{\partial^2 F}{\partial t^2}(x,t) = c^2 \frac{\partial^2 F}{\partial x^2}(x,t) \\ F(0,t) = F(\pi,t) = 0 \end{cases} \qquad (\textit{Vibrating string equation})$$

5) Find A_n such that $\begin{cases} \frac{\partial^2 F}{\partial t^2}(x,t) = c^2 \frac{\partial^2 F}{\partial x^2}(x,t) \text{ on }]0,\pi[\times]0,T[\\ F(0,t) = F(\pi,t) = 0 \text{ for } t \in [0,T[\\ F(x,0) = x(\pi - x) \text{ for } x \in [0,\pi] \end{cases}$ where T is some positive real number,

assuming that the necessary term by term differentiations are still valid on $]0,\pi[\times]0,T[$ in this case.

<u>Exercise 3</u> (~ 5-6 points)

Consider the following differential equation (*Airy equation*):

$$(\text{E}): \; y''(x) + x\, y(x) = 0$$

It is a useful equation in physics and although it looks very simple, it is difficult to solve it.

1) Suppose that y is a solution of (E) in an open interval of center 0.
What is the value of $y''(0)$?

2)
We will admit that (E) has a unique solution f such that $f(0)=1$ *and* $f'(0)=0$.
We are going to show that f can be expanded in a power series in $\mathbb{R}$ *and use this expansion to get an approximation of f.*

Suppose that $f(x)=\sum_{n=0}^{+\infty} a_n x^n$ and suppose that f is a solution of (E) on $]-R,R[$ such that $f(0)=1$ and $f'(0)=0$.

a) Find the inductive relationships between the coefficients a_n.
b) Prove that $\forall p \in \mathbb{N}, a_{3p+1}=0, a_{3p+2}=0$
c) For $p\in\mathbb{N}$, calculate a_{3p} in terms of p and give the radius of convergence R of the series you have obtained.
d) *Rapid study of the approximation of f(x) given by* $S_n(x)=\sum_{p=0}^{n} a_{3p}x^{3p}$ *on* [0,1]:

Prove that $\forall n\in\mathbb{N}, \forall x\in[0,1], \left|f(x)-S_n(x)\right| \le \left|a_{3n+3}x^{3n+3}\right|$.

Deduce that $\forall x\in[0,1], \left|f(x)-S_2(x)\right| \le \frac{1}{2\times3\times5\times6\times8\times9} = \frac{1}{12960}$.

SCAN **Test of Mathematics (3)** **March 26, 2009**

(1 h 30)

No documents, no calculators, no mobile phones allowed
Clearly state the theorems you use.

Exercise 1 (~ 5.5 points)

1) Let m be a given real number.

If $\vec{u}=(x,y)\in \mathbb{R}^2$, put $q_m(\vec{u}) = x^2 + y^2 + 2m\,xy$; q_m defines a quadratic form of $\mathbb{R}^2$.

Write the matrix A_m of q_m in the standard basis $\mathcal{B}$ of $\mathbb{R}^2$.

For which values of m is q_m definite positive?

Write q_m as an algebraic sum of squares in a direct orthonormal basis $\mathcal{B}'$ of $\mathbb{R}^2$ for its usual dot product.

2) Let $(\mathcal{C}_m)$ be the curve of the xy-plane of equation: $x^2 + y^2 + 2m\,xy = 4$ in the direct orthonormal set of axes (O; $\mathcal{B}$).

Write the equation of $(\mathcal{C}_m)$ in (O; $\mathcal{B}'$) and recognize its nature according to the values of m.

Plot $(\mathcal{C}_3)$ (in the initial set of axes (O; $\mathcal{B}$) of the xy-plane).

Exercise 2 (~ 4.5 points)

Consider the following partial differential equation:

(E) $y\dfrac{\partial^2 f}{\partial x\partial y} - x\dfrac{\partial^2 f}{\partial x^2} - \dfrac{\partial f}{\partial x} = y$ with f: U $\subset \mathbb{R}^2 \to \mathbb{R}$, a C^2 function.

1) Show that the change of variables ϕ defined by $\phi(x,y)=(u,v)$ with $u = xy$ and $v = y$ defines a C^2 diffeomorphism between $U = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$ and $V = \{(u,v) \in \mathbb{R}^2 \mid v > 0\}$ (remark: $U = V$.)

2) Put f(x,y) = F(u,v) = F(xy,y) in $U = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$.

Write the partial differential equation that is checked by F in V and solve it.

Deduce the solutions of (E) in $U = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$.

Exercise 3 (~ 2 points)

Consider the following change of variables $\begin{cases} x = uv \\ y = u^2 + v^2 \end{cases}$.

(T*he study of this change of variables is not required*)

Let f be a C^2 real valued function in $\mathbb{R}^2$ and put $F(u,v) = f(uv,u^2+v^2)$.

Using the notations $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ for the partial derivatives of f and the notations $\dfrac{\partial F}{\partial u}$, $\dfrac{\partial F}{\partial v}$ for the partial derivatives of F, express $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at the point (x,y) in terms of the partial derivatives of F at the point (u,v).

Exercise 4 (~ 9 points)

The questions 2 and 3 are completely independent.

Suppose that a and b are 2 given real numbers with $a < b$.
Call E = $\mathcal{C}([a,b], \mathbb{R})$, the set of continuous functions from $[a,b]$ into $\mathbb{R}$.
Let $w:[a,b] \to \mathbb{R}$ be a given continuous function that is strictly positive on $[a,b]$.
Put $< f, g >= \int_a^b f(t)g(t)w(t)dt$.

1) Prove that $<.,.>$ defines a scalar product of E.

2) For this part and only in this part, assume that $a = -1, b = 1, \forall t \in [-1,1], w(t) = 1$ and let e_1, e_2, and e_3 be the 3 functions defined by $e_1(t) = 1$, $e_2(t) = t$ and $e_3(t) = t^2$ for $t \in [-1,1]$.
Call F the subspace of E that is spanned by these 3 functions *i.e.* $F = \{a_1e_1 + a_2e_2 + a_3e_3 \mid a_1, a_2, a_3 \in \mathbb{R}\}$.
Admit that $\mathcal{B} = (e_1, e_2, e_3)$ is a basis of F.
a) Write the matrix A of $<.,.>$ in the basis $\mathcal{B} = (e_1, e_2, e_3)$.
b) Apply the Schmidt process to $\mathcal{B}$ to find an orthonomal basis $\mathcal{B}_1=(f_1,f_2,f_3)$ of $(F, <.,.>)$.
c) Let f be the function of E defined by $f(t) = t^3$.
Find the orthogonal projection (versus $<.,.>$) of f onto F and deduce $\min_{(\alpha,\beta,\gamma)\in\mathbb{R}^3} \int_{-1}^1 (t^3 - \alpha - \beta t - \gamma t^2)^2 dt$.

3) Back to the general case:
Call $E_2 = \mathcal{C}^2([a,b], \mathbb{R})$, the subspace of E made up of the C^2 functions of E.
Let p and q be two given C^1 functions of E.
We are interested in the functions y belonging to E_2 that satisfy the conditions $(\mathcal{P}_\lambda)$ for some real number λ with $(\mathcal{P}_\lambda)$: $\begin{cases} \forall t \in [a,b], (p(t)y'(t))' + q(t)y(t) + \lambda w(t)y(t) = 0 \\ y(a) = y(b) = 0 \end{cases}$.

Let Ψ be the linear operator defined on E_2 by $\Psi(f) = \frac{1}{w}\left((p\, f')' + q\, f\right)$.
a) Say why a solution y of $(\mathcal{P}_\lambda)$, for some real number λ, is an eigenvector of Ψ.
b) Verify that $(p\, f')'g = (p\, f'g - p\, f\, g')' + (p\, g')'f$ for all f and g in E_2.
Deduce that if $f \in E_2$, $g \in E_2$ and if $f(a) = f(b) = g(a) = g(b) = 0$ then
$$< \Psi(f), g >=< f, \Psi(g) >.$$
c) Deduce that if y_1 is a solution of $(\mathcal{P}_{\lambda 1})$ and if y_2 is a solution of $(\mathcal{P}_{\lambda 2})$ with $\lambda_1 \neq \lambda_2$ then $< y_1, y_2 >= 0$.

SCAN **Test of Mathematics (4)** **May 7, 2009**

(1h 30)

No documents, no calculators, no mobile phones allowed

***Extrema of functions of several variables* (~ 7-8 points)**

Exercise 1

1) Find the relative extrema on $\mathbb{R}^2$ of the function *f* that is defined by $f(x,y)=12xy-3xy^2-x^3$.

2) Say why *f* has absolute extrema in on $\Delta=[0,1]\times[0,1]$ and find their values.

3) Does *f* have absolute extrema on $\mathbb{R}^2$? (*Justify your answer*)

Exercise 2

Let *a*, *b* and *c* be given positive(>0) real parameters; the volume of the domain limited by the surface (S) : $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is given by $V(a,b,c)=\frac{4}{3}\pi abc$

The lengths *a*, *b* and *c* are to vary so that *a*+*b*+*c* = *K* where *K* is a given positive constant.

How should you choose *a*, *b* and *c* to get a volume that is maximal*?*

***Implicit functions* (~ 9 points)**

Exercise 3

The 2 parts are independent of each other.

I) Let (γ) be the curve of the *xy*-plane of implicit equation $x^3y+y^3x+x-y+1=0$.

1) Show that in the neighbourhood of *A*(0,1), the curve (γ) has a local representation of the form $y=\varphi(x)$.

2) Say why φ is C^∞ in the neighbourhood of 0.

Give the values of $\varphi(0)$, $\varphi'(0)$ and $\varphi''(0)$.

3) Give the equation of the tangent line to (γ) at *A* and give the position of the curve with respect to the tangent line around *A*. Draw the curve (γ) and its tangent line in the neighbourhood of *A*.

II) More generally:

Let ($\mathcal{C}$) be the curve of the *xy*-plane of implicit equation *F*(*x*,*y*)=0 where *F* is a given C^2 function from $\mathbb{R}^2$ into $\mathbb{R}$.

We will denote F'_x, F'_y, F''_{xx}, F''_{xy}, F''_{yy} the partial derivatives of order 1 or 2 of *F* versus its first or second variable ($F'_x=\frac{\partial F}{\partial x}, F'_y=\frac{\partial F}{\partial y}, F''_{xx}=\frac{\partial^2 F}{\partial x^2}, F''_{xy}=\frac{\partial^2 F}{\partial x\partial y}, F''_{yy}=\frac{\partial^2 F}{\partial y^2}$)

Let $M_0(x_0,y_0)$ be a given point of ($\mathcal{C}$) such that $\nabla F(x_0,y_0)\neq(0,0)$. For simplicity we will assume that $F'_y(x_0,y_0)\neq 0$

1) Write the equation of the tangent line to ($\mathcal{C}$) at $M_0(x_0,y_0)$.

2) We remind you that for a curve (Γ) of equation $y=f(x)$, with f that is C^2 in the neighbourhood of x_0, if the curve (Γ) has an inflection point at x_0 then necessarily $f''(x_0)=0$.

Here, prove that if $M_0(x_0,y_0)$ is an inflection point of ($\mathcal{C}$) then necessarily,

$$F''_{xx}\times\left(F'_y\right)^2-2F''_{xy}\times F'_x\times F'_y+F''_{yy}\times\left(F'_x\right)^2=0 \text{ at } M_0(x_0,y_0).$$

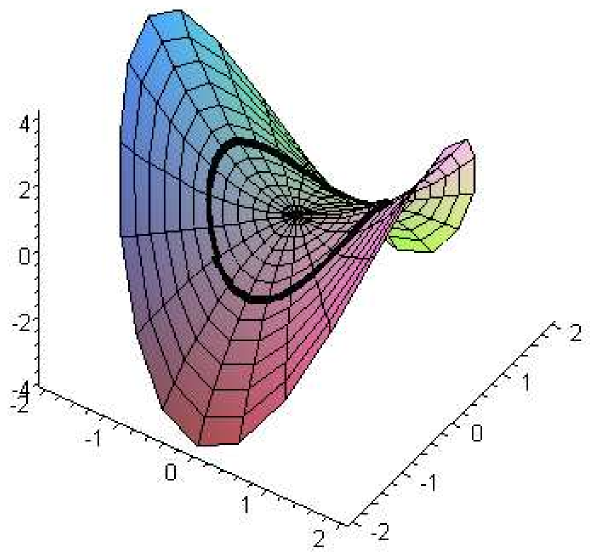
Surfaces (~3-4 points)

Exercise 4

Let (Σ) be the surface of parametric representation $\begin{cases} x = \lambda \cos\theta \\ y = \lambda \sin\theta \\ z = \lambda^2 \cos 2\theta \end{cases} \quad (\lambda, \theta) \in R \times [0, 2\pi]$.

1) Find the singular points of this parametrized sheet.
2) Give a normal vector $\vec{N}_0$ to (Σ) at the point $M_0(1,0,1)$ and write the equation of the tangent plane to (Σ) at M_0.
3) Give an implicit equation of (Σ).
4) Express by means of an integral that you will not try to evaluate, the length of the coordinate line $\lambda = 1$.

```
> with(plots) : ds := plot3d([l·cos(t), l·sin(t), l^2·cos(2·t)], t = 0 .. 2
      ·Pi, l = −2 ..2) :
  dc := spacecurve([cos(t), sin(t), cos(2·t)], t = 0 .. 2·Pi, thickness
      = 3, color = black) :
  display(ds, dc);
```



SCAN 2 — Test #1 — 1h30 *November 12, 2009*

No document, no calculator, no mobile-phone allowed.

Exercise 1. All questions are independant.

1. Show that a continuous linear map is differentiable. What is its differential?
2. Show that the function
$$f(x,y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \dfrac{x^2y^2}{x^2+y^2} & \text{otherwise.} \end{cases}$$
is continuous on $\mathbb{R}^2$.
3. Recall the definition of equivalent norms.
4. Show that $N(x,y) = |x| + |2y|$ defines a norm on $\mathbb{R}^2$ and sketch the unit ball.

Exercise 2. Let $\omega \in \mathbb{R}_+^*$. Solve
$$y'' + 4y = \sin \omega t.$$

Exercise 3. Give the real form of solutions of the linear differential system
$$\begin{cases} x' = -x + 2y \\ y' = z \\ z' = 2x - 3y + 2z. \end{cases}$$
Deduce the real solutions of
$$\begin{cases} x' = -x + 2y \\ y'' - 2y' = 2x - 3y \\ x(0) = 1,\ y(0) = y'(0) = 0. \end{cases}$$

Exercise 4. In this exercise we study differential equations of the form

(E) $$y'(x) = g\left(\frac{y}{x}\right)$$

on an interval I with $0 \notin I$, where g is a continuous function.

1. Set $u(x) = \dfrac{y(x)}{x}$. Show that (E) is equivalent to:

   (E') $$u'(x) = \big(g\big(u(x)\big) - u(x)\big)\frac{1}{x}.$$

   How are the equilibrium solutions of (E') determined?
2. Consider the following problem:

   (P) $$\begin{cases} y'(x) = \dfrac{x+2y}{2x+y} \\ y(1) = 0. \end{cases}$$

   Explain why problem (P) has a unique solution in an open interval containing 1.
3. Show that (P) can be written as:
$$\begin{cases} y'(x) = g\left(\dfrac{y}{x}\right) \\ y(1) = 0. \end{cases}$$
   for a function g you will explicit.
4. Give the equation of the solution curve of (P) in this case (don't try to explicit the solution).

No document, no calculator, no mobile-phone allowed.

Exercise 1. In this exercise, all questions are independant.

1. What is a geometric series? When does it converge? When does it diverge? When it converges, what is its sum?
2. Let $(u_n)_{n\geq 0}$ be a sequence of real numbers. For all $n \in \mathbb{N}$ we set $v_n = u_{n+1} - u_n$. Under what condition is the series
$$\sum_{n=0}^{\infty} v_n$$
convergent? Compute its value in this case.
3. Let g a function of class C^2 on $\mathbb{R}$ and let $f(x, y) = g\left(\frac{y^2}{x}\right)$. Where is the function f of class C^2? Compute $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$.
4. Where do the following relations define a local diffeomorphism?
$$\begin{cases} x = u^3 + v^3 \\ y = uv. \end{cases}$$
Compute $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$.

Exercise 2. We consider the following partial differential equations:

$$(\mathrm{E}_0) \qquad x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = 0,$$

$$(\mathrm{E}_1) \qquad x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} + x^2 f(x, y) = 0.$$

1. Show that the following change of variables can be used to solve (E_0) and (E_1) on $U = \mathbb{R}_+^* \times \mathbb{R}_+^*$:
$$\begin{cases} s = \frac{x}{y} \\ t = y. \end{cases}$$
2. Find all the functions of class C^2 on U that solve Equation (E_0).
3. Find all the functions of class C^2 on U that solve Equation (E_1) as well as the following initial conditions:
$$f(\pi, y) = 0 \text{ and } f\left(\frac{\pi}{2}, y\right) = y, \ \forall y \in \mathbb{R}_+^*.$$

Exercise 3. Let P and Q be polynomials with real coefficients and ε a real number. If a is a root of P and if ε is small enough, it is legitimate to conjecture that $P + \varepsilon Q$ will have a root $\varphi(\varepsilon)$ close to a. The goal of this exercise is to study the roots of the perturbed polynomial $P + \varepsilon Q$ on a particular example.
We set $P = X^2 - 4$, $Q = X^3$, and $a \in \{-2, 2\}$ is a root of P. We also set

$$f(x, \varepsilon) = (x^2 - 4) + \varepsilon x^3.$$

1. Apply the implicit function theorem to the function f at the point $(a, 0)$ and show that the equation

   $$(x^2 - 4) + \varepsilon x^3 = 0 \qquad (1)$$

   defines x in terms of ε in a neighborhood of $(a, 0)$. We denote by $x_a(\varepsilon)$ this solution.
2. Of what differentiability class is the function x_a?
3. Give the Taylor expansion of order 2 of $x_a(\varepsilon)$ in a neighborhood of 0 for $a = -2$ and $a = 2$.
4. Say why, for ε sufficiently close to 0 but not 0, the three solutions of Equation (1) are real. We denote by $x_3(\varepsilon)$ the third root.
5. Show that
   $$x_{-2}(\varepsilon) + x_2(\varepsilon) + x_3(\varepsilon) = -\frac{1}{\varepsilon}.$$
6. Deduce a generalized expansion of order 2 at 0 of $x_3(\varepsilon)$.
7. Explain why for $\varepsilon > 0$ small enough, the three roots of Equation (1) are in the following order:
   $$x_3(\varepsilon) < x_{-2}(\varepsilon) < x_2(\varepsilon).$$
   In what order are the roots of Equation (1) if $\varepsilon < 0$ and close enough to 0?

SCAN 2 — Test #3 — 1h30 *March 25, 2010*

No document, no calculator, no mobile-phone allowed.

Exercise 1. We consider the following series of functions:

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n + n^2 x^2}.$$

1. Give the domain (in $\mathbb{R}$) of the function f.
2. Show that for all $a > 0$ the series f converges uniformly on $[a, +\infty)$. Deduce that f is continuous on $(0, +\infty)$.
3. Without trying to differentiate it, show that the function f is decreasing on $(0, \infty)$.
4. Show that f can be differentiated term by term on $(0, +\infty)$.
5. Find a constant A such that
$$\forall x > 0,\ f(x) \leq \frac{A}{x^2}.$$
Deduce the value of the limit $\lim_{x \to +\infty} f(x)$.
6. Prove that for all $x > 0$,
$$f(x) \geq \int_1^{+\infty} \frac{dt}{t(1 + tx^2)} = \ln\left(1 + \frac{1}{x^2}\right).$$
Deduce the value of the limit $\lim_{x \to 0^+} f(x)$.

Exercise 2. We recall that the notation $C\big([0,1]\big)$ stands for the set of real-valued continuous functions on $[0,1]$.
Let $\alpha \in \mathbb{R}$. The goal of this exercise is to study the solutions of the following differential problem:

$$\text{(D)} \qquad \begin{cases} \forall x \in [0,1],\ f'(x) = \sin\big(xf(x)\big) \\ f(0) = \alpha. \end{cases}$$

1. Preliminary question: Show that for all real numbers a and b, the following inequality holds true:
$$|\sin b - \sin a| \leq |b - a|.$$
2. Find an integral equation in $C\big([0,1]\big)$ that is equivalent to System (D).
3. Use the Fixed Point Theorem to show that the integral equation from the previous question has a unique solution in $C\big([0,1]\big)$.
4. Explicit a sequence of functions (f_n) that approaches uniformly the solution f of System (D). Give an estimate of $\|f - f_n\|_\infty$. Compute $f_2(x)$ when $\alpha = 1$ and $\forall x \in [0,1],\ f_0(x) = 0$.

Exercise 3. In this exercise, the two parts are independant.

Part I

Let $A \in M_n(\mathbb{C})$ and B an element of $\mathbb{C}^n$.

1. Use the Fixed Point Theorem to show that if there is a subordinate matrix norm such that $|||A||| < 1$ then the linear system

$$X = AX + B \tag{1}$$

of unknown X has a unique solution in $\mathbb{C}^n$, and explicit a sequence of elements of $\mathbb{C}^n$ that converges to the fixed point.

2. Show that $\rho(A) \leq |||A|||$, where $\rho(A)$ is the spectral radius of A defined as:

$$\rho(A) = \max\big\{|\lambda|;\ \lambda \in \mathbb{C} \text{ is an eignenvalue of } A\big\}.$$

3. In this question only we study two special cases:

   a) In this question only we consider

$$A = \frac{1}{8}\begin{pmatrix} -4 & 1 & 1 \\ -1 & 1 & 2 \\ 2 & -2 & 1 \end{pmatrix}.$$

   Show that System (1) as a unique fixed point and that the sequence you defined in Question 1 converges.

   b) In this question only we consider

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

   Show that we cannot apply the method from Question 1 in this case.

Part II

In this part we assume that $A \in M_n(\mathbb{C})$ is a diagonalizable matrix, that is, there exists an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

1. Show that if $\|\cdot\|$ is a norm on $\mathbb{C}^n$ then $\|\cdot\|_P$ defined as

$$\forall X \in \mathbb{C}^n,\ \|X\|_P = \big\|P^{-1}X\big\|$$

is also a norm on $\mathbb{C}^n$.

2. We denote by $|||\cdot|||_P$ the matrix norm subordinate to $\|\cdot\|_P$. Show that $|||A|||_P = |||D|||$.
3. Deduce that there exists a subordinate norm N such that $N(A) = \rho(A)$.

SCAN 2 — Test #4 — 1h30 *May 7, 2010*

No document, no calculator, no mobile-phone allowed.

Exercise 1.

1. Give the definition of a scalar product.
2. In the vector space $\mathbb{R}^2$ we define the following quadratic form:

$$q(x,y) = x^2 + y^2 + xy.$$

   a) Give the matrix associated with the quadratic form q in the canonical basis of $\mathbb{R}^2$.

   b) We consider the following basis of $\mathbb{R}^2$:

$$\mathcal{B} = \big((1,1),(1,-1)\big).$$

   Give the matrix associated with the quadratic form q in the basis $\mathcal{B}$.

   c) Compute $\varphi\big((1,1),(1,-1)\big)$, where φ is the polar form associated with the quadratic form q.

Exercise 2.

1. Give the radius of convergence of the power series

$$\sum_{n=1}^{+\infty} n x^n$$

   and compute its sum.

2. We consider the following rational function:

$$\varphi(x) = \frac{x}{(1-x)^2\big(1+x^2\big)}.$$

   Give the power series expansion of φ, and specify the domain of validity.

3. We want to solve the following system of sequences, for $n \in \mathbb{N}$:

$$\begin{cases} a_{n+1} + b_n = n \\ a_n - b_{n+1} = 1 \\ a_0 \in \mathbb{R},\ b_0 = -1. \end{cases}$$

   We consider the following power series:

$$f(x) = \sum_{n=0}^{+\infty} a_n x^n.$$

   a) Show that

$$f(x) = \frac{a_0 + x(1-2a_0) + a_0 x^2}{\big(1+x^2\big)(1-x)^2}.$$

   b) In the case $a_0 = 0$, deduce an expression of a_n and b_n in terms of n.

Exercise 3. Let f be the function defined by:

$$f(x) = 1 + \pi - \frac{8}{\pi} \sum_{n=0}^{+\infty} \frac{\cos(2n+1)x}{(2n+1)^2}.$$

1. a) Show that f is continuous on $\mathbb{R}$.
   b) Show, using Fourier series, that for all $x \in [0, \pi]$,

$$f(x) = 2x + 1.$$

   c) Sketch the graph of the function f on the interval $[-2\pi, 2\pi]$.
   d) Give the value of the sum of the series:

$$\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2}.$$

2. Let $n \in \mathbb{N}$ and $\alpha_n \in \mathbb{R}$. We consider the following linear differential equation

$$(E_n) \qquad y'' + 4y' + y = \alpha_n \cos nt.$$

   a) Show that the function y_n defined by

$$\forall t \in \mathbb{R},\ y_n(t) = A_n \cos nt + B_n \sin nt$$

   is a solution of Equation (E_n) if

$$A_n = \frac{\alpha_n\big(1-n^2\big)}{\big(1-n^2\big)^2 + 16n^2}, \qquad B_n = \frac{4n\alpha_n}{\big(1-n^2\big)^2 + 16n^2}.$$

   b) Give a simple equivalent of A_n and B_n when $n \to +\infty$.
   c) Let $(\alpha_n)_{n\geq 0}$ be a sequence of real numbers such that the series $\sum |\alpha_n|$ converges. Deduce from the previous question a particular solution of the following linear differential equation:

$$y'' + 4y' + y = \sum_{n=0}^{+\infty} \alpha_n \cos nt.$$

   d) Deduce the solutions of the following linear differential equation:

$$y'' + 4y' + y = f.$$