SCAN **Test of Mathematics (2)** **December 18, 2008**

(1 h 30)

No documents, no calculators, no mobile phones allowed
Clearly state the theorems you use.

Exercise 1 **(~2 points)**

Give the radius of convergence and find the sum on its open interval of convergence of the following power series: $\sum_{n=0}^{+\infty}(-1)^n nx^n$. Deduce the value of the following sum: $s = \sum_{n=0}^{+\infty}(-1)^n \frac{n}{2^n}$.

Exercise 2 **(~12-13 points)**

Part I results can be used without proof to answer Part II.

Part I (~7-8 points)

Consider the **odd** periodic function f of **period 2π** such that $f(x) = x(\pi - x)$ in $[0,\pi]$.

1) Plot its curve on $[-3\pi,3\pi]$. (*Don't forget that f is odd)*

2) Calculate the Fourier series of *f*.

3) Deduce that for all x in $\mathbb{R}$, $f(x) = \frac{8}{\pi}\sum_{k=0}^{+\infty}\frac{1}{(2k+1)^3}\sin((2k+1)x)$. Clearly justify your answer.

4) Deduce the value of the following sums $s_1 = \sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k+1)^3}$ and $s_2 = \sum_{k=0}^{+\infty}\frac{1}{(2k+1)^6}$.

Part II (~5-6 points)

In this part, c is a given positive real number.

Suppose that $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers such that

$$\exists K \in \mathbb{R}^+, \exists \alpha > 1, \forall n \in \mathbb{N}^*, |A_n| \le \frac{K}{n^\alpha}.$$

For $x\in\mathbb{R}$ and $t\in\mathbb{R}$, put $F(x,t) = \sum_{n=1}^{+\infty} A_n\cos(nct)\sin(nx) = \sum_{n=1}^{+\infty} u_n(x,t)$ with $u_n(x,t) = A_n\cos(nct)\sin(nx)$

1) Prove that the series defining F is convergent for all $x\in\mathbb{R}$ and $t\in\mathbb{R}$.

For a given t, we will consider the function of x defined by $g_t(x) = F(x,t)$.

For a given x, we may also consider the function of t defined by $h_x(t) = F(x,t)$.

2) Prove that g_t is continuous on $\mathbb{R}$ (t is fixed). What do you get for h_x?

3) Prove that if $\alpha > 3$, g_t is C^2 on $\mathbb{R}$ and that it can be twice differentiated term by term.
What do you get for h_x?

4) Deduce that if $\alpha > 3$, F is a solution on $[0,\pi]\times\mathbb{R}$ of the following problem:

$$\begin{cases} \frac{\partial^2 F}{\partial t^2}(x,t) = c^2\frac{\partial^2 F}{\partial x^2}(x,t) \\ F(0,t) = F(\pi,t) = 0 \end{cases} \qquad (\textit{Vibrating string equation})$$

5) Find A_n such that $\begin{cases} \frac{\partial^2 F}{\partial t^2}(x,t) = c^2\frac{\partial^2 F}{\partial x^2}(x,t) \text{ on }]0,\pi[\times]0,T[\\ F(0,t) = F(\pi,t) = 0 \text{ for } t\in[0,T[\\ F(x,0) = x(\pi - x) \text{ for } x\in[0,\pi] \end{cases}$ where T is some positive real number,

assuming that the necessary term by term differentiations are still valid on $]0,\pi[\times]0,T[$ in this case.

Exercise 3 **(~ 5-6 points)**

Consider the following differential equation (*Airy equation*):

$$\text{(E)}: \; y''(x) + x\, y(x) = 0$$

It is a useful equation in physics and although it looks very simple, it is difficult to solve it.

1) Suppose that y is a solution of (E) in an open interval of center 0.
What is the value of $y''(0)$?

2)
We will admit that (E) has a unique solution f such that $f(0)=1$ *and* $f'(0)=0$.
We are going to show that f can be expanded in a power series in $\mathbb{R}$ *and use this expansion to get an approximation of f.*

Suppose that $f(x)=\sum_{n=0}^{+\infty} a_n x^n$ and suppose that f is a solution of (E) on $]-R,R[$ such that $f(0)=1$ and $f'(0)=0$.

a) Find the inductive relationships between the coefficients a_n.

b) Prove that $\forall p \in \mathbb{N}, a_{3p+1}=0, a_{3p+2}=0$

c) For $p \in \mathbb{N}$, calculate a_{3p} in terms of p and give the radius of convergence R of the series you have obtained.

d) *Rapid study of the approximation of f(x) given by* $S_n(x)=\sum_{p=0}^{n} a_{3p} x^{3p}$ *on* [0,1]:

Prove that $\forall n \in \mathbb{N}, \forall x \in [0,1], \left| f(x)-S_n(x) \right| \le \left| a_{3n+3} x^{3n+3} \right|$.

Deduce that $\forall x \in [0,1], \left| f(x)-S_2(x) \right| \le \dfrac{1}{2\times3\times5\times6\times8\times9} = \dfrac{1}{12960}$.