Exercice 1

$u_n = (-1)^n n x^n$ $\qquad \left|\frac{u_{n+1}}{u_n}\right| = \frac{n+1}{n}|x| \to |x|$ $\qquad$ therefore $R = 1$

on $]-1,1[$ $\qquad \sum_{n=0}^{\infty} (-1)^n n x^n = \sum_{n=1}^{\infty} (-1)^n n x^n$ $\quad$ if $n x^n = 0$ if $n=0$

$= x \sum_{n=1}^{\infty} (-1)^n n x^{n-1} = x \left(\sum_{n=0}^{\infty} (-1)^n x^n\right)'$ (the sum of a power series can be differentiable term by term in its open interval of cv)

(geom. series of ratio $-x$) $\leftarrow$ $= x\left(\frac{1}{1+x}\right)' = -\frac{x}{(1+x)^2}$

$x = \frac{1}{2} \in]-1,1[$, hence $\sum_{n=0}^{\infty} (-1)^n \frac{n}{2^n} = -\frac{1}{2}\frac{1}{(1+\frac{1}{2})^2} = -\frac{2}{9}$

Exercice 2 (part I)

1) on $[0,\pi]$ $\quad f'(x) = \pi - 2x$

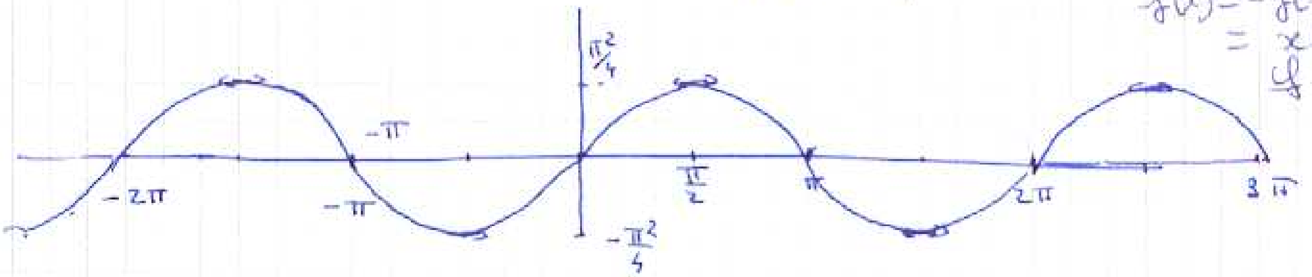
x	0		$\pi/2$		π
f'		$+$	0	$-$	
f	0	$\nearrow$	$\frac{\pi^2}{4}$	$\searrow$	0

Rk: On $[-\pi,0]$

$f(x) = -f(-x) = -(-x)(\pi-(-x))$

$= x(x+\pi)$

if $-x \in [0,\pi]$



2) f is odd $\leadsto a_n = 0 \quad \forall n \in \mathbb{N}$ (including a_0)

$T = 2\pi \qquad \omega = \frac{2\pi}{T} = 1$

$\forall n \geq 1 \qquad b_n = \frac{2}{T}\int_{\alpha}^{\alpha+T} f(t) \sin nt \, dt$

$= \frac{2}{T}\int_{-\pi}^{\pi} f(t) \sin nt \, dt$ $\qquad$ ($t \mapsto f(t)\sin nt$ is even) product of odd f^{ns}

$= \frac{4}{T}\int_0^{\pi} f(t) \sin nt \, dt$

$= \frac{2}{\pi}\int_0^{\pi} t(\pi - t) \sin nt \, dt$.

by parts $\qquad u = t(\pi - t) \qquad u' = \pi - 2t$

$v' = \sin nt \qquad v = -\frac{1}{n}\cos nt$.

$b_n = \frac{2}{\pi}\left(\left[-\frac{t}{n}(\pi-t)\cos nt\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}(\pi-2t)\cos nt\, dt\right)$

$= \frac{2}{\pi n}\int_0^{\pi}(\pi - 2t)\cos nt \, dt$

$u = \pi - 2t \qquad u' = -2$
$v' = \cos nt \qquad v = \frac{1}{n}\sin nt$

$b_n = \frac{2}{\pi n}\left(\left[\frac{\pi-2t}{n}\sin nt\right]_0^{\pi} + \frac{2}{n}\int_0^{\pi}\sin nt \, dt\right) = \frac{4}{\pi n^2}\int_0^{\pi}\sin nt \, dt$

$$b_n = \frac{4}{\pi n^2}\left[-\frac{\cos nt}{n}\right]_0^{\pi} = \frac{4}{\pi n^3}\left[1-(-1)^n\right]$$

$$= \begin{cases} 0 & \text{if } n = 2p \\ \dfrac{8}{\pi(2p+1)^2} & \text{if } n = 2p+1 \end{cases}$$

Concl: $S(f)(x) = \frac{8}{\pi}\sum_{p=0}^{\infty} \frac{\sin(2p+1)x}{(2p+1)^3}$

3) Use Dirichlet
- f is periodic of period 2π
- f is piecewise smooth over a period $[-\pi,\pi]$
  cf it is C^1 on $[-\pi,\pi]$ except at 3 points $-\pi, 0, \pi$ where f and f' have finite limits on the right and on the left (restriction to $[-\pi,0]$ and $[0,\pi]$ of C^1 $f(x)$)
- f is continuous on $\mathbb{R}$

therefore $\forall x \in \mathbb{R}$ $f(x) = S(f)(x) = \frac{8}{\pi}\sum_{k=0}^{\infty}\frac{\sin(2k+1)x}{(2k+1)^3}$

4) - $x = \frac{\pi}{2}$ $\quad \sin(2k+1)\frac{\pi}{2} = (-1)^k$

$$f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4} = \frac{8}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3} \rightsquigarrow S_1 = \frac{\pi^3}{32}.$$

- Parseval:

$$\frac{2}{T}\int_{\alpha}^{\alpha+T} f^2(x)\,dx = 2a_0^2 + \sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

$$\underset{\text{here}}{=} \sum_{n=1}^{\infty} b_n^2 = \sum_{k=1}^{\infty}\frac{64}{\pi^2}\frac{1}{(2k+1)^6}.$$

$$\rightsquigarrow S_2 = \frac{\pi^2}{64} \times \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \frac{\pi^2}{64}\,\frac{2}{\pi}\int_0^{\pi} x^2(\pi-x)^2\,dx.$$

$$= \frac{\pi}{32}\int_0^{\pi} \pi^2x^2 + x^4 - 2\pi x^3\,dx.$$

$$= \frac{\pi}{32}\left[\pi^2\frac{\pi^3}{3} + \frac{\pi^5}{5} - 2\frac{\pi^5}{4}\right] = \frac{\pi^6}{32}\left(\frac{1}{3}+\frac{1}{5}-\frac{1}{2}\right)$$

$$= \frac{\pi^6}{32\times 30} = \frac{\pi^6}{960}.$$

not in the test

- $\sum_{n=0}^{\infty}\frac{1}{n^6} = \sum_{p=0}^{\infty}\left(\frac{1}{(2p)^6}+\frac{1}{(2p+1)^6}\right)$ (we can group 2 by 2 cf $\frac{1}{n^6}\to 0$)

$$= \frac{1}{2^6}\sum_{p=0}^{\infty}\frac{1}{p^6} + \sum_{p=0}^{\infty}\frac{1}{(2p+1)^6}$$

$$\rightsquigarrow \left(1-\frac{1}{2^6}\right)\sum_{p=0}^{\infty}\frac{1}{p^6} = s_2$$

$$s_3 = s_2\,\frac{2^6}{2^6-1}$$

Part II

1) $|u_n(x,t)| \leq |A_n| \leq \frac{K}{n^\alpha}$.

as $\alpha > 1$, $\sum \frac{K}{n^\alpha}$ cv and therefore $\sum u_n(x,t)$ cv (AC).

2) • $v_n(x) = u_n(x,t)$ is continuous on $\mathbb{R}$ versus x for all n

• $\sup_{x \in \mathbb{R}} |v_n(x)| \leq \frac{K}{n^\alpha}$ and $\sum \frac{K}{n^\alpha}$ cv $(\alpha > 1)$

therefore $\sum v_n$ is uniformly cv on $\mathbb{R}$ (Ω test).

$\rightsquigarrow$ g_t is continuous on $\mathbb{R}$.

. for the same reasons h_x is continuous on $\mathbb{R}$.

3) Use the term by term differentiation theorem.

• $\forall n$, v_n is C^2 on $\mathbb{R}$.

. $\exists x_0 \in \mathbb{R}$ s.t $\sum v_n(x_0)$ cv (cf cv on $\mathbb{R}$)

. $\sum v_n'$ and $\sum v_n''$ are ucv on $\mathbb{R}$ via the Ω-test:

cf • $\sup_{x \in \mathbb{R}} |v_n'(x)| = \sup_{x \in \mathbb{R}} |n A_n \cos(nct) \cos nx| \leq \frac{nK}{n^\alpha} = \frac{K}{n^{\alpha-1}}$

and $\sum \frac{K}{n^{\alpha-1}}$ cv $(\alpha > 3)$

• $\sup_{x \in \mathbb{R}} |v_n''(x)| = \sup_{x \in \mathbb{R}} |-n^2 A_n \cos nct \sin nx| \leq \frac{n^2}{n^\alpha} K = \frac{K}{n^{\alpha-2}}$

and $\sum \frac{K}{n^{\alpha-2}}$ cv $(\alpha > 3)$.

Cond: g_t is C^2 on $\mathbb{R}$ and can be twice diff. term by term.

. for the same reasons, this result is valid for h_x.

4) • $\frac{\partial^2 F}{\partial t^2} = \sum_{n=1}^{\infty} -n^2 c^2 A_n \cos nct \sin nx$ cf 3)

$\frac{\partial^2 F}{\partial x^2} = \sum_{n=1}^{\infty} -n^2 A_n \cos nct \sin nx$.

$\rightsquigarrow$ $\frac{\partial^2 F}{\partial t^2} = c^2 \frac{\partial^2 F}{\partial x^2}$.

• $F(0,t) = \sum_{n=1}^{\infty} A_n \cos nct \times 0 = 0 = F(\pi, t)$

$\sin 0 = \sin n\pi = 0$.

5) $F(x,0) = \sum_{n=1}^{\infty} A_n \sin nx = x(\pi - x)$ on $]0, \pi[$

$\rightsquigarrow$ $A_n = 0$ if $n = 2p$

$A_n = \frac{8}{\pi} \frac{1}{(2p+1)^2}$ if $n = 2p+1$.

Exercise 3

1) $y''(0) = -0 \times y(0) = 0$.

2) in $]-R, R[$ $\quad f(x) = \sum_{n=0}^{\infty} a_n x^n$

$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

(a power series can be differentiated term by term in its open interval of convergence)

Hence, in $]-R, R[$,

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0.$$

$$\leadsto \begin{cases} 2a_2 = 0 \\ \forall n \geq 1 \quad (n+2)(n+1) a_{n+2} + a_{n-1} = 0. \end{cases}$$

(uniqueness of the expansion of the nil function in $]-R, R[$)

$$\forall n \geq 1 \quad a_{n+2} = -\frac{a_{n-1}}{(n+2)(n+1)}$$

b) $a_0 \leadsto a_3 \leadsto a_6 \leadsto \cdots \leadsto a_{3p}$.

$a_1 \leadsto \cdots a_{3p+1}$

$a_2 \leadsto \cdots a_{3p+2}$.

. as $a_2 = 0$, $a_{3p+2} = 0$ $\forall p \in \mathbb{N}$. (obvious induction)

. $a_1 = \frac{f'(0)}{1!} = 0$ so $a_{3p+1} = 0$ $\forall p \in \mathbb{N}$. (obvious induction)

c) $$\left.\begin{aligned} a_{3p} &= -\frac{a_{3p-3}}{3p \times 3p-1} \\ a_{3p-3} &= -\frac{a_{3p-6}}{3p-3 \times 3p-4} \\ &\vdots \\ a_6 &= -\frac{a_3}{6 \times 5} \\ a_3 &= -\frac{a_0}{3 \times 2} \end{aligned}\right\} \leadsto a_{3p} = (-1)^p \frac{a_0}{2 \times 3 \times 5 \times 6 \times 8 \times 9 \times \cdots \times (3p-1) \times 3p}.$$

with $a_0 = f(0) = 1$.

Radius of cv: $y(x) = \sum_{p=0}^{\infty} \underbrace{a_{3p} x^{3p}}_{u_p}$

$$\left|\frac{u_{p+1}}{u_p}\right| = \left|\frac{a_{3p+3}}{a_{3p}}\right| |x|^3 = \frac{1}{(3p+3)(3p+2)} |x|^3 \longrightarrow 0 < 1 \quad \forall x$$

so $\boxed{R = +\infty}$

d) for $x \in [0,1]$, the series checks the alternating series test for convergence of $u_p = (-1)^p |u_p|$, $u_p \to 0$ and $|u_{p+1}| \leq |u_p|$ (cf $\left|\frac{u_{p+1}}{u_p}\right| \leq 1$)

therefore : $|f(x) - S_n(x)| = |R_n(x)| \leq |u_{n+1}| = |a_{3n+3} x^{3n+3}|$

$$\forall x \in [0,1] \quad |f(x) - S_2(x)| \leq |a_9 \times 1^9| = |a_9| = \frac{1}{2 \times 3 \times 5 \times 6 \times 8 \times 9} \quad //$$